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AN

ELEMENTARY TREATISE

ON

MECHANICS:

INTENDED FOR THE USE OF COLLEGES
AND UNIVERSITIES.

BY

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WITH EXTENSIVE CORRECTIONS AND ADDITIONS.

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PREFACE TO THE SEVENTH EDITION.

In the successive editions of my *Elementary Treatise on Mechanics*, I have introduced such alterations as appeared likely to give a more philosophical character to the study of the subject, without disturbing the student by too extensive changes in the form of the work. In the Sixth Edition, the innovations were, perhaps, inconveniently large, including omissions of portions of the previous editions. In the present edition, I have restored some of these omitted portions, especially the *Mechanical Powers;* and I hope that the work will thus still be found useful to the class of students who used the previous editions.

I may, however, remark that the subjects omitted in the Sixth Edition were contained in another volume, which I published about the same time, the *Mechanics of Engineering,* and which may be looked upon as a *Supplement* to the *Elementary Treatise on Mechanics.* In that volume, I gave a number of investigations concerning *Machines,* treated in a manner more complete and philosophical than the old arrangement of the *Mechanical Powers* allowed;—concerning *the Principle of Virtual Velocities,* of which Principle a complete proof cannot be given, without a complete classification of Machines;—concerning *Statical Couples;*—concerning *Structures,* as *Frames* with *Ties, Struts* and *Braces;*—concerning *Arches,* both *Direct* and *Oblique;*—concerning the *Equilibrium of Machines with Friction,* treated in a more correct and complete manner than in the former editions of the *Elementary Treatise;*—concerning *Stable and Unstable Equilibrium;*—concerning the *Elasticity and Flexure of Solid Materials;*—concerning *Vis Viva,* and

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Prof. Alex. Ziwet
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the measure of *Labouring Force,* in which part the recent views of Poncelet and other French engineers were briefly but systematically given. To this I added some investigations concerning *Impact,* principally borrowed from Mr Airy and M. Poncelet. The *Elementary Treatise* and the *Mechanics of Engineering* taken together, made up, I conceive, a much more complete Treatise than any of the former editions of my work.

The reason for dividing the work into these two parts was, as I then stated, the wish to give effect to the separation of the two sciences of *Mechanics* and *Mechanism;* a separation to which various writers, as well as myself, had been led, as a mode of making the subject more philosophical than had previously been usual. Professor Willis, in his *Treatise on Mechanism,* published about the same time, gave us a far more complete and better arranged work than we possessed before, on the modes and properties of *Motion* transferred by mechanism; and I was thus led to dwell upon the conditions which govern the *Force,* when motion is thus transferred. I still hope to see this division of the subject, and the scientific cultivation of each part of it, make great advances among our students of mathematics.

The present volume contains the more elementary parts of Mechanics, given, I hope, in a simple and logical manner. It appears to me highly desirable, in Mechanics as in Geometry, to preserve the early and historical form of the more elementary propositions. It is in dealing with these historical forms, that the mechanical ideas of speculative minds have from age to age been unfolded into clearness and distinctness: and a knowledge of these forms of reasoning is therefore highly valuable, as giving the student a share in the historical progress which the mind of man has made, in these lines of speculation. This advantage is lost, if we adopt some new mode of beginning the study of the subject, even if the new mode be more philosophical than the old; which its novelty may often make it seem to be,

when really it is not so. In order to secure this advantage to our students, there ought to be a *Permanent Portion* of the subject of Mechanics, which shall not change with the changing views of each race of mathematicians: nor will the existence of such a Permanent Portion of the subject impede the cultivation of the Progressive Portion of the science, any more than the study of Euclid impedes the study of Monge and Legendre. In this Permanent Portion of Mechanics, we must, I conceive, begin our reasonings concerning Forces with the Lever, and deduce the other properties, as for example the Composition of Forces, from the Lever; we must prove the properties of the Center of Gravity in particular and simple cases, before we proceed to the general case; we must prove the Laws of Falling Bodies, of Projectiles, and of Oscillating Pendulums, by particular methods of summation, detached from the General Methods of the Differential Calculus, and from any other mode of treating Central Forces in general; since all General Methods belong to a stage of intellectual discipline for which the Permanent and Elementary Portions of Science are only a preparation. I have accordingly taken the course I have thus described; and I am persuaded that we shall very fatally damage the effect of our mathematical studies as an intellectual discipline, if we ever adopt a course in which we disregard these long accepted proofs, and urge our students to rush, from the first, into General Methods, and to occupy their minds with novel modes of contemplating elementary truths.

TRIN. COLL. *May* 28, 1847.

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AN ELEMENTARY TREATISE

ON

MECHANICS.

INTRODUCTION.

1. *Mechanics is a science which treats of the motion and rest of bodies as produced by Force.*

Force is any cause which moves or tends to move a body, or which changes or tends to change its motion.

Every science involves certain ideas, by means of which we give unity and connection to our sensations, and in virtue of which we are able to reason concerning the facts which we perceive by our senses. Thus Geometry involves the idea of Space, Arithmetic, the idea of Number; and conditions resulting from the nature of Space and of Number are applicable to all the objects of our external experience. In like manner, the science of Mechanics involves the idea of Cause; which idea, when applied to the facts of motion and equilibrium, gives rise to the conception of Force.

The appearances and occurrences of the material world suggest to us the conception of motion, and of changes of motion. Moreover, we find that we can often, by our own volition and exertion, influence the motions of bodies, and occasion changes of motion. We perceive too, that bodies appear to influence each other's motion in the same manner. By considering these occurrences in a general and abstract manner we obtain the conception of *Force*. Force is conceived as that general and abstract property by which one body causes, changes, or prevents motion in another body.

Thus, when a man supports a stone in his hand, his hand is said to exert force upon the stone: and in the same manner, if he move a machine by turning a winch, he is said to exert force on the winch, and, by this, on the machine. If the machine be moved by the weight of a heavy body, this heavy

body is said to exert force. When a stone falls, it is said to be moved by the force of gravity, or of the Earth's attraction.

2. *Body or Matter is any thing extended and possessing the power of resisting the action of force.*

In the conception of force exerted, there is involved the notion of a certain power of resistance, residing in the object on which the force is exerted. This power of resistance shews itself by the object excluding other bodies from the space it occupies; by its transmitting force to other bodies; by its requiring a larger force to produce a quicker motion; and in other ways.

In this manner the solid bodies which are treated of in Mechanics differ from the solids treated of in Geometry. In the last-mentioned science we conceive figures to possess extension only, without tangible solidity; they are mere modifications and limitations of our notion of space; they occupy space without excluding other figures from the same space; they have no material substance or mechanical attributes. In Mechanics we consider bodies as they really exist; not only as extended, but as impenetrable, stiff or flexible, inert, heavy.

3. The difference of the kinds of extension treated of in Geometry and in Mechanics is sometimes expressed by calling the lines and planes which we have to do with in Mechanics, *material* or *physical* lines and planes. Thus a physical line is a linear body, as a fine wire or rod, of which we do not say, as in geometry, that it has length, and no breadth or thickness; but of which we do not consider the breadth and thickness in our reasoning, although we suppose the line to have rigidity, weight, and inertia, which without breadth and thickness it could not have. And in the same manner, a material plane has weight and inertia, and may have rigidity. A material point or particle also has weight and inertia, even although we do not consider its extension. A material solid has weight, inertia, impenetrability, none of which are attributes of mere geometrical solids. It may also have either complete rigidity, or flexibility with a greater or less amount of elasticity.

4. Material lines may be considered as *made up* of material points; material planes, as made up of material points or of

material lines; material solids as made up of material points, or lines or planes. The thickness of the planes, the breadth of the lines, the length of the points thus introduced as elements, is never considered as finite; and our reasonings are always so conducted, that in the sequel these dimensions disappear; but in the course of our reasonings, we must needs conceive material bodies thus made up of material elements.

5. We cannot conceive force to be exerted without conceiving matter as that on which and by which it acts. Thus if we push at a stone with a staff, we exert force upon the stone by means of the properties which belong to the staff as matter—its rigidity and coherence. And if we exert force by means of the hand alone, we no less produce the effect by means of the material instrument which we use, namely, the hand itself.

6. We conceive force as necessarily acting *on* matter, but not as necessarily residing in matter and acting by means of matter. The pressure or the fall of a heavy body is conceived as produced by the force of gravity, this force not residing in any material vehicle, or operating on the body by material contact; but being an immaterial influence,—a mere attraction. And in like manner any other attraction, as that of a magnet on iron, is conceived as an immaterial influence, producing in the body pressure or motion, as material pressure or impulse might do.

There have been controversies whether these attractions of gravity and magnetism are or are not really transmitted by some material substance which acts upon bodies directly by contact: but however these controversies be decided, it is often necessary, in the science of Mechanics, to *conceive* attractions as immaterial influences, which, when they act upon matter, become sensible to us as force.

7. *Bodies are capable of motion of various kinds.*

We possess the conception of motion, which we constantly see exemplified among the bodies which we observe about us. These change their places and positions by moving in various ways; as when a stone falls to the ground, or a heavy body slides down a slope, or a carriage is overturned. And these cases may exemplify three different kinds of motion. The

first is, the motion of a small body, which we consider as a particle or material *point;* and in which we do not consider any change in the relative position of the parts of the body. In this motion of a point, the *path* may be any curve whatever. A second kind of motion is, when a body of finite magnitude slides, *retaining its parallelism;* that is, all lines in the body remaining parallel to their first position, and all points in it describing parallel lines. A third kind of motion is, when a body turns *about an axis;* the axis remaining fixed, at least for an instant, and the points of the body moving round the axis. In this case the path of each point of the body is an arc of a circle, in a plane perpendicular to the axis. The motion is called a motion of *rotation;* and is contrasted with a motion of *translation*, by which the whole body is transferred from one place to another.

We have now to consider some of the other properties which we must necessarily conceive as belonging to force.

8. *Forces may produce either rest or motion in bodies.*

A single force acting upon a body necessarily produces motion; but two or more forces may be combined so as to destroy each other's effects and produce rest. Thus one person pulling at a rope which is free, moves it; but two persons, pulling with equal strength at the two ends of a rope, will not produce motion: they will exactly *balance* each other's effects, and produce rest. If a person standing on the bank of a canal pull a floating boat by means of a rope, it will move towards him. But if several persons, some on one bank and some on the other, pull the boat, each towards himself, the strength which they employ, and the directions in which they pull, may be so adjusted, that the forces exerted may exactly balance each other, and the boat may remain at rest. In all such cases forces are said to *balance* each other, or to be *in equilibrium.* They are also said to *counteract*, or to *destroy* each other.

9. *The science of Mechanics is divided into two parts, Statics and Dynamics.*

When any forces act upon any body or system of bodies, there are, as has just been stated, two possible cases respecting

the result: the forces may be exactly such as to counteract or balance each other,—such as to produce equilibrium; or they may produce motion. In the latter case, we have to consider not only the forces, but the direction, velocity, and duration of the motion; in the former case, we have only to consider the relations of the forces which thus balance each other. Hence these two cases may most conveniently be treated of separately.

We must, therefore, divide our subject into these two parts,—

STATICS, which treats of forces in equilibrium:

DYNAMICS, which treats of forces producing motion.

10. *Forces are measurable quantities.*

All causes must be measured by their effects; and force, as a conception included in the idea of cause, must be measured by the effects which it produces. But in Mechanics, we consider different kinds of effects of force: as equilibrium, velocity, momentum, and others; and the measure of force will be different according to the kind of effect which we contemplate. Hence the measures of force will be different in Statics and in Dynamics; and also, as we shall see, different in different parts of Dynamics.

11. The reasonings of each branch of Mechanics must depend upon the mode of measuring force in that branch. But in all the branches, the forces, being measurable, will be capable of addition, and consequently of numerical expression. For by the assumption of an arbitrary unit, all magnitudes may be produced by addition; and may be expressed by reference to the unit.

12. *In forces we have to consider the direction and point of application*, as well as the magnitude. The direction of a force is the direction in which it tends to produce motion. The point of application is the point at which it is applied or exerted.

Forces are *directly opposite* when they act in the same line but in opposite directions.

13. *Reaction is equal and opposite to action.*

In all material agency and causation we may conceive a reaction of the same kind as the action. If we press a stone with the hand, it presses the hand in return; if we strike it, we receive a blow by the act of giving one; if we urge it so as to give it motion, we lose some of the motion which by the same effort we should give to our limbs if the stone did not impede them. In all these cases there is a reaction of the same kind as the action; and, this being the case, the reaction must necessarily be opposite to the action, and equal to it. For the action and the reaction may each be conceived as determining the other; they are mutually cause and effect; and, therefore, depend each upon the other by the same law. And, therefore, there can be no reason why one should be greater or less than the other, or in a different line. They are necessarily equal and opposite.

But though we thus perceive that, of whatever kind mechanical action be, it must be accompanied with an equal and opposite reaction, we cannot apply this axiom without knowing what action is, and how it is measured. Now mechanical action, or force, as we have just seen, admits of various measures according to the effect contemplated. In Statics, the action is pressure; in Dynamics, it depends upon the velocity, and size, and other circumstances of the body moved. In both these cases, we assert reaction to be equal and opposite to action; but the assertion receives very different interpretations in the two cases.

14. *In Statics a Force is measured by the Force which directly balances it.*

In all cases reaction is equal and opposite to action. In Statics, the action is the force which produces equilibrium, and the reaction is a force of the same kind; hence these two are equal, and each measures the other.

Thus, if a man sustain any heavy body in his hand, the force with which the heavy body tends downwards, measures the force which the man exerts with his hand. If a man pulling one way can exactly counteract two boys pulling the opposite way, the force of the man is equal to that of the two boys.

15. *All bodies have weight.*

All bodies which come under our notice fall or tend to fall downwards. This force may be totally counteracted, in which case the body will rest, producing pressure on its support; as a stone held in the hand: or the force with which the body tends to fall may not be counteracted, the support being withdrawn, and then the body moves downwards.

This fall or pressure of bodies downwards is conceived to arise from the universal influence of a certain abstract force, *gravity*, or the *attraction* of the Earth. This force makes bodies *heavy;* and heavy bodies have weight.

The term *weight* is employed above to designate a *property* by which bodies are heavy: but the same term is employed to denote the heavy *body* itself, considered with reference to a measure of heaviness. Thus we speak of a weight of one hundred pounds, meaning a body which weighs one hundred pounds.

16. *The weights of bodies are measurable quantities.*

Weight is considered with reference to its mechanical effects; and two bodies which produce the same effect have equal weights. Thus, if a given heavy body A, hung to a spring, draw it out to a certain distance, and if another heavy body B, hung in the same manner, draw the spring out to the same distance, (A being removed,) B and A have equal weights. Also, if C draw the spring out as far as A and B do when hung on together, C is equal to the sum of A and B, and is therefore double of A. And if D draw out the spring as far as C and A together do, D is three times A; and thus we have a measure of weights.

We here speak of a spring balance; but if any other mechanical effect were taken, the process of constructing a scale of measurement of weights, would be the same: if, for instance, P, in a common pair of scales, balance W, and if Q in the same pair of scales also balance W, P is equal to Q.

If P be one pound, Q will also be one pound, C two pounds, D three pounds; and in the same manner for any other unit of weight.

17. *In Statics, forces are measured by the weights which they can support.*

For to support a weight is to balance it, and we have already seen (Art. 14) that forces are measured by the forces which they balance.

Thus if a man supports five pounds in his hand, his hand exerts a force of five pounds. A rope to which a "hundredweight" is appended, exerts a force of one hundred and twelve pounds.

18. *The weight of a body is the sum of the weights of its parts.*

By joining bodies together we have their weights; and the weight of the whole is independent of the arrangement of the parts. A basket of stones is of the same weight, however the stones are shaken into new positions. The same lump of material is of the same weight, into whatever new form it be moulded. If $2\frac{1}{2}$ cubic inches of lead be one pound, 25 cubic inches of lead will be ten pounds, whatever be its shape.

19. *The quantities of matter of bodies are measured by their weights.*

Matter is apprehended by its mechanical effects. The quantity of matter is conceived to be the *same*, when the mechanical effect is the same; and in order to measure a quantity of matter, we must take some property of matter which admits of addition, so that the amount of the property in the whole mass is the same as the sum of its amounts in the several parts. Since weight is such a property, we may measure the quantities of matter of bodies by their weight.

20. *The densities of different kinds of matter are different.*

The same effect may be produced by masses of different magnitudes, when the materials are different. Thus $3\frac{1}{2}$ cubic inches of iron will produce the same effect by its weight as $2\frac{1}{2}$ cubic inches of lead. Hence the quantity of matter in $3\frac{1}{2}$ cubic inches of iron is the same as the quantity of matter in $2\frac{1}{2}$ cubic

inches of lead. This difference of magnitude is conceived to arise from this; that such materials consist of small particles with hollow spaces intervening, and that in lead these particles are closer together, or *denser*, than in iron, and the intervening spaces, or pores, less. Hence lead has a greater density than iron; and the same is applicable to other cases.

21. *Quantity of matter is also measured by inertia.*

Matter is originally apprehended, as we have said, by its resistance to the action of force. When we put a mass in motion, we find that it resists; and it resists the more, and receives less motion, as it is larger. This is expressed by ascribing to the body a property of *inertia;* in virtue of which it resists, or does not fully yield to, the action of force. And as this inertia increases with the mass of the body, and is thus capable of addition, we may take it as a measure of the quantity of matter.

When we come to establish the Third Law of Motion, we shall find that inertia, so understood, is as the weight; and hence this measure of quantity of matter agrees with the one already given in Art. 19.

22. *Bodies transmit force in various ways, according as they are rigid or flexible, free or constrained.*

Matter, as we have seen, is apprehended in virtue of its *resisting* force; but while it resists, it *transmits* force. We push a stone with the hand, and the stone resists the pressure; but it transmits the pressure to some contiguous body, as for instance, the ground; and if there be no such contiguous body, the stone does not resist the pressure, but yields to it, and is put in motion. And the ways in which we can thus transmit force are different in different kinds of bodies. If we have a weight fastened at the end of a *staff* which we hold, we can either push the weight from us, or pull it to us, by means of the staff: but if the weight be fastened at the end of a *cord*, of which we hold the other end, we can pull the weight towards us, but we cannot by means of the cord push it from us. If we rest the middle of the staff on our knee, we can, by pressing down one end, raise the weight fastened at the other; but we

can produce no such effect with the cord. These differences arise from the staff being *rigid*, while the cord is *flexible*. Again, in the latter case, in which the staff rests upon a point and turns round that point, the point is a *fulcrum* or fixed point; or, considering the body with reference to space of three dimensions, the staff turns round an *axis*. A rigid rod thus turning round an axis and transmitting force by that means is a *lever*. And the two portions intervening between the fulcrum and the force, at the one end or at the other, are the *arms* of the lever. Again, a flexible cord may pass round a peg, or a fixt *pully*, and may thus pass into a direction different from that which it had before it reached the pully: and it will transmit in this new direction the force which acted upon it on the other side of the pully.

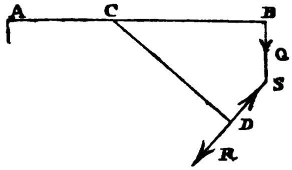
23. Forces exerted by means of any solid bodies are called *Pressures*. A rigid body transmitting pressure is in a state of *compression*. Cords which transmit force are in a state of *tension;* and this tension is measured by the force which they transmit.

24. *A Pressure transmitted directly* in the straight line in which it acts, is transmitted without augmentation or diminution, whether by means of a rigid or a flexible body. If we push by means of a rod, if we pull by means of a cord, the force transmitted to the farther extremity of the rod or the cord is exactly the same as that which we exert at the end which we hold. In stating this, we of course leave out of consideration the weight of the rod and of the cord.

That matter thus transmits force directly, without altering its amount, is a necessary consequence of this equality of action and reaction. When a force which acts at A in the direction of the straight line AB, transmits a force to the point B, it may be counteracted by an opposite force at B, acting in the direction BA. These opposite forces at A and at B which thus balance each other must necessarily be equal; for each may be considered as the reaction with reference to the other.

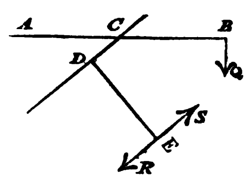
25. *A pressure transmitted round a fulcrum* by means of a lever is transmitted without augmentation or diminution, if the arm of the lever be supposed to turn through

any angle; the lever, thus bent, still continuing rigid. Thus the lever *ACB*, moveable about the fulcrum *C* in the plane *CDQ*, and acted upon by a force *Q* perpendicular to *CB*, tending to turn the lever round *C* in the direction *BQ*, transmits to another point *D* a force, also tending to turn the lever round *C*, if *D* be rigidly connected with *CB*. And if *CD* be equal to *CB*, the force which is transmitted to *D* will be a force *R*, perpendicular to *CD*, and equal to *Q*. This also appears by the equality of action and reaction. For, *CD* being equal to *CB*, if the action of the force *Q* be balanced by a reaction *S*, perpendicular to *CD*, *S* will be equal to *Q*, for each may be considered as the reaction with reference to the other. And since the force *Q* requires to be balanced by a force *S* equal to *Q*, *Q* must be equal to the force *R*; for *R* is directly balanced by *S*, and is therefore equal to it.



A lever *ACD* in which the arms are not in the same straight line is a *bent lever*.

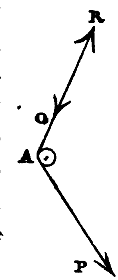
26. A *pressure transmitted round any axis* is transmitted without augmentation or diminution, if the arm by which it is transmitted be supposed to be turned through any angle, and transferred any distance along the axis. Let *ACB* be a lever moveable about an axis *CD*, and acted upon by a force *Q*, perpendicular to *CB*, tending to turn the lever round *CD* in the direction *BQ*, (*CD* being perpendicular to the plane *CBQ*;) and let *DE* be the arm *CB* transferred to *D* along the axis and turned through any angle. The force *Q* transmits to *E* a force tending to turn the lever round the axis *CD*, if *DE* be rigidly connected with *CB*. And if *DE* be equal to *CB*, the force which is transmitted to *E* will be a force *R*, perpendicular to *DE* and equal to *Q*. This appears by the equality of action and reaction. For *DE* being equal to *CB*, if the action of the force *Q* be balanced by a reaction *S*, perpendicular to *CD*, *S* will be



equal to Q, because each of these may be considered as the reaction with reference to the other. And since the force Q requires to be balanced by a force S equal to it, Q must be equal to R; for R is directly balanced by S, and is therefore equal to it.

27. *A pressure transmitted by a cord*, round a peg, or a pully, is transmitted without augmentation or diminution.

Thus a force P is transmitted round the pully A to Q, and produces at Q a force in the direction QA, equal in amount to the force P in the direction AP. For the force P may be balanced by force R, acting in AQ; and R will be equal to P, since each is the reaction with respect to the other. But Q is directly balanced by R, and is therefore equal to R. Hence P and Q are equal.



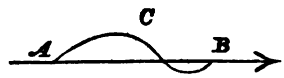
Here we leave out of consideration the weight of the cord, and any difficulty which arises in the motion of the cord round the fixed obstacle A. A *pully* is a contrivance (a roller upon an axis) to make the cord to run round the fixed point A without resistance.

28. A rigid body is one in which the relative positions of the parts are not capable of any change, in consequence of force applied to them. Such a body, as we have seen, transmits force from one point to another: and it has appeared that it transmits force either directly in a straight line; or round a fulcrum in one plane, as in a lever; or round an axis.

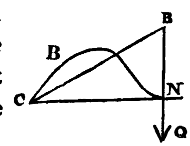
In order to transmit force in this manner, the rigid body must be capable of resisting *compression* and *extension*, *flexure* and *tension*.

29. *The shape of a rigid body may be conceived to be altered in any manner*, and no alteration will be required in the forces; provided the fulcrum and the point of application of the force remain the same. For the only effect of the rigidity of the body is to transmit the force from one point of the body to another; and this will be the consequence of a rigid connection

of the parts, whatever be the form. Thus instead of pressing the point B with a straight rod AB, we may push it with a crooked stick ACB; but still in the same direction



AB, and the effect will still be the same, of transmitting the force unchanged from A to B. The same would be true, if CB were the arm of a lever: however the arm CBN be bent, the effect of the force Q is the same as if it acted at a straight arm CN. For the effect of the force Q is to turn the plane CBN about C; and the only effect of the rigidity of the plane is to make a statical connection between



the centre of motion C and the point of application N. In the same manner in the case of a body moveable about an axis, the form of the body makes no difference in the force transmitted. And therefore any body moveable about an axis, and acted upon by forces tending to turn it round the axis, may be reasoned of as if it were a *wheel and axle*, that is, a rigid machine of particular form, moveable about an axis.

Hence it is not at all necessary when a rigid body transmits force, that the lines in which we suppose the forces to act should lie within the body.

30. We have spoken of bodies as absolutely rigid, or absolutely flexible. In actual fact, however, no bodies are so rigid as not to be in some degree capable of change of form; and no bodies so flexible as not to offer some resistance to flexure. The *compressibility of materials*, and the *rigidity of cords*, exist in almost all cases; but they are at present left out of consideration. They are taken account of in certain ulterior parts of the science of Mechanics, of which we shall not treat in the present Work.

31. *We may suppose immoveable fulcrums and immoveable axes*, about which levers and rigid bodies are moveable. The forces which act upon the levers and bodies produce pressure upon these fulcrums; and we suppose the fulcrums capable of supplying all the reaction which is needed for equilibrium, whatever its amount may be. This is what we mean by calling the fulcrums immoveable.

32. Hence we may at any point of a system in equilibrium, substitute for an immoveable fulcrum the force which the fulcrum exerts; and in like manner, for any force, we may substitute a fulcrum at the same point. The pressure *of* the fulcrum and the pressure *on* the fulcrum are always equal forces in opposite directions.

33. *We may also suppose immoveable surfaces*, against which bodies are pressed by forces. Thus when a body rests on the ground, we may suppose the ground to react with such a force, whatever it may be, as is requisite for equilibrium. When a body is supported on an inclined plane, we may suppose the inclined plane to exert a force, which, along with the other forces that act, keeps the body in equilibrium.

34. We can conceive the surface against which a body is pressed to be perfectly hard, and perfectly smooth. On this supposition the surface would offer no resistance to the motion of a body in any direction upon it; and the reaction of the plane must necessarily be perpendicular to the surface.

35. But if the surface be not perfectly smooth, it will offer a resistance to a body moving along it. This resistance is called *friction*, and is a force which impedes the motion of a body. But the same force which impedes the motion when it has begun, will, within certain limits at least, prevent the motion before it begins. The defect of perfect smoothness in an inclined board, will make a book slide down it more slowly than it otherwise would; but also, the same defect will prevent the book from sliding altogether, when the inclination of the board to the horizon is very small. Whereas if the board be perfectly smooth, the smallest inclination of the board will make the book begin to slide.

The force arising from this defect of absolute smoothness, is termed *friction*, because it arises from the rubbing which takes place when the body moves. But if the body be prevented from moving, it sticks, and the friction has sometimes, by way of explanation, been called *stiction*. When a body is pressed against any surface, the effect of this force may be represented by conceiving, along with the reaction which is perpendicular to the surface, another force which

acts along the surface, in a direction determined by the direction of the motion, or of the tendency to motion, of the body so pressed against the surface.

36. If a flexible body be in equilibrium, any part of it may be supposed to become rigid without disturbing the equilibrium. For since the body is in equilibrium, its parts have no tendency to turn about each other. Hence in this case, the force of rigidity, which is only the power of resisting such a tendency, is not called into play. There is no tendency to change of figure in the flexible body, and therefore it makes no difference whether such a tendency would be resisted if it did exist; that is, it makes no difference, as to the forces which keep any part in equilibrium, whether it be supposed rigid or not.

We now proceed to establish the propositions of Statics; the fundamental doctrines which we have here explained being first re-stated as Definitions and Axioms.

STATICS.

BOOK I. ELEMENTARY PROPOSITIONS.

CHAPTER I. THE LEVER.

Art. 37. Definitions. I. Forces are *directly opposite* when they act in the same line in opposite directions. (Introduction, Art. 12.)

II. Two directly opposite forces which balance each other are equal. (Int. 14.)

III. Forces are capable of *addition*. (Int. 11.)

IV. A force is *twice* as great as a given force, when it is the sum of two others each equal to the given force; a force is *three* times as great, when it is the sum of three such forces; and so on. (Int. 11.)

V. Forces in Statics may be *measured* by the weights they could support. (Int. 14.)

VI. A *Lever* is a rigid rod, moveable in one plane about a point which is called the *fulcrum* or *centre of motion*, acted on by forces which tend to turn it round the fulcrum. (Int. 22.)

VII. The portions of the rod between the fulcrum and the points where the forces are applied are called the *arms*. (Int. 22.)

VIII. When the arms are two portions of the same straight line, the line is called a *straight* lever; otherwise, it is called a *bent* lever. (Int. 25.)

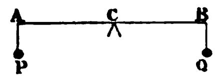
IX. The Lever is supposed to be without weight, except the contrary be expressed.

X. If forces, or lines, or both, be expressed by numbers, the product of a force acting upon a line, multiplied into the perpendicular drawn from the centre of motion, upon the direction of the force, is called the *moment* of the force.

38. Axioms. Axiom I. If two equal forces act perpendicularly at the extremities of equal arms of a straight lever

to turn it opposite ways, they will keep each other in equilibrium.

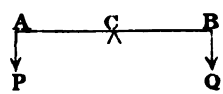
If $AC = BC$, and P and Q be two equal forces acting perpendicularly on CA and CB, at A and B, they will balance.



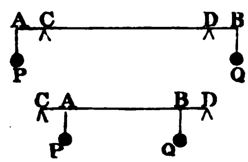
II. If forces keep each other in equilibrium, and if any force be added to one of them, it will preponderate.

III. If two equal weights balance each other upon a horizontal straight lever, the pressure upon the fulcrum is equal to the sum of the weights, whatever be the length of the lever. (Int. 18.)

If P, Q be two equal weights which balance each other upon the horizontal lever AB, the pressure upon C is $P + Q$.



IV. If two equal weights be supported upon a straight lever on two fulcrums, at equal distances from the weights, the pressures upon the two fulcrums are together equal to the sum of the weights. (Int. 18.)

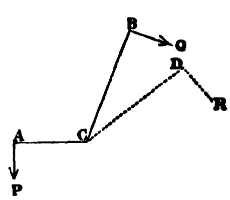


If P, Q be two equal weights which are supported upon the line AB on two fulcrums C, D, so that AC, BD are equal; the pressures upon C, D are together equal to the sum of the weights $P + Q$.

V. On the same suppositions, the pressures on the two fulcrums are equal.

VI. If a force act perpendicularly on the straight arm of a bent lever at its extremity, the effect to turn the lever round the fulcrum will be the same, whatever be the angle which the arm makes with the other arm, so long as the length is the same. (Int. 25.)

If a force Q act perpendicularly on CB at its extremity B, C being the fulcrum, and an equal force R act perpendicularly on an equal arm CD, at its extremity, the effect to turn the lever round C in the two cases is equal.



VII. When a force acts upon a rigid body it will produce the same effect to urge the body in the line of its own direction, at whatever point of the direction it acts. (Int. 24.)

VIII. If a body which is moveable about an axis be acted upon by two equal forces, in two planes perpendicular to the axis, the forces being perpendicular at the extremities of two straight arms of equal length from the axis; the two forces will produce equal effects to turn the body, at whatever points the arms meet the axis. (Int. 26.)

IX. If a string pass freely round a fixed body, so that the direction of the string is altered, any force exerted at one extremity of the string will produce at the other extremity the same effect as if the force had acted directly. (Int. 27.)

X. If in a system which is in equilibrium, there be substituted for the force acting at any point, an immoveable fulcrum at that point, the equilibrium will not be disturbed. (Int. 32.)

XI. If in a system which is in equilibrium, there be substituted for an immoveable point or fulcrum, the force which the fulcrum exerts, the equilibrium will not be disturbed. (Int. 32.)

XII. A perfectly hard and smooth surface, acted on at any point, by any force, exerts a reaction which is perpendicular to the surface at that point; and if the surface be supposed to be immoveable, the force will be supported, whatever be its magnitude. (Int. 34.)

XIII. A heavy material straight line, prism or cylinder, of uniform density, may be supposed to be composed of a row of heavy points of equal weight, uniformly distributed along the line.

XIV. A heavy material plane of uniform density may be supposed to be composed of a collection of parallel straight lines of equal density, uniformly distributed along the plane.

XV. A heavy solid body of uniform density may be supposed to be composed of a collection of particles, the weight

of each of which is as the portion of the body which it occupies, and which may be considered as heavy points.

XVI. If a flexible body be kept in equilibrium by any forces, any portion of it may be supposed to become rigid, after the equilibrium is established, and the forces which keep the system in equilibrium will not be altered.

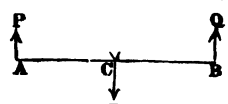
39. POSTULATES. I. A prism or cylinder of uniform density, and of given length, may be taken, which is equal to any given weight.

II. A force may be taken equal to the excess of a greater given force over a less.

III. A force may be taken in a given ratio to a given force.

40. PROP. *If a weight be supported on a horizontal rod by two forces acting vertically at equal distances from the weight, the forces are equal to each other, and their sum is equal to the weight.*

Let the two forces P, Q act perpendicularly at the extremities of the equal arms CA, CB of the horizontal lever AB; and let them balance each other. The forces P, Q will be equal; for if not, let one of them, as P, be the less, and by Post. II. take X, the force which is the excess of Q above P, so that $P + X$ is equal to Q; therefore, by Ax. I, $P + X$ will balance Q. But since P balances Q, if we add to P the force X it will preponderate, by Ax. II; which is absurd. Therefore P is not less than Q; and in the same manner it may be shewn that Q is not less than P. Therefore P and Q are equal.

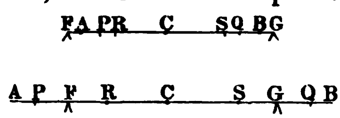


And since P and Q are equal, by Ax. III, the pressure on the fulcrum C is equal to the sum of the two forces P, Q. Hence, by Ax. XI, if, instead of a fulcrum, there be a force R, acting at C perpendicularly to the lever, and equal to the sum of P and Q, this force will balance the pressure at C, just as the fulcrum does, and there will still be equilibrium; that is, a vertical force or weight R will be supported by two forces P, Q, acting vertically at equal distances CA, CB; and

these forces are equal; and the weight R is equal to the sum of P and Q. Q. E. D.

41. PROP. *A horizontal prism or cylinder of uniform density will produce the same effect by its weight as if it were collected at its middle point.*

Let AB be the prism or cylinder, and C its middle point. Let P, R be any points in AC, and let CQ be taken equal to CP, and CS equal to CR.



The half AC of the prism may (by Ax. XIII.) be supposed to be made up of small equal weights, distributed along the whole of the line AC, as at P, R; and the half BC may in like manner be conceived to be made up of small equal weights distributed along BC, as at Q, S; CQ being equal to CP, CS to CR, and so on.

Let F be a fulcrum about which the prism AB tends to turn by its weight. In CB, produced if necessary, take CG equal to CF, and suppose a fulcrum placed at G.

Let the weights at P, Q, R, S be denoted by P, Q, R, S.

The two weights P and Q produce upon the fulcrums F and G pressures which together are equal to the sum of the weights $P + Q$ (Ax. IV), or to the double of P, since P and Q are equal; hence (Ax. V.) the pressure upon each of them is P; therefore the pressure upon the fulcrum G, arising from the two weights P and Q, is P; in like manner the pressure upon the fulcrum G, arising from R and S, is R; and so of the rest: and the whole pressure on G, arising from the whole prism AB, is the sum of all the weights P, R, &c. from A to C; that is, it is half the weight of the prism.

But if the whole prism be collected in its middle point C, the pressure upon the two fulcrums F and G will be the whole weight of the prism; and the pressures on the two fulcrums are equal, by Art. 40. Therefore, in this case also, the pressure on the fulcrum G is equal to half the weight of the prism. Therefore the prism, when collected at its middle point, produces the same pressure on the fulcrum G as it did before.

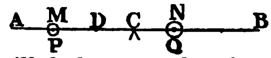
Therefore, when a uniform prism is collected at its middle point, it produces the same effect by its weight as it did before. Q. E. D.

Cor. 1. A uniform prism or cylinder will balance itself upon its middle point.

Cor. 2. When a prism or cylinder thus balances upon its middle point, the pressure upon the fulcrum is equal to the weight of the prism.

42. Prop. *If two weights act perpendicularly on a straight lever on opposite sides of the fulcrum, and if the weights are inversely as the arms of the lever, they will balance each other.*

Let P, Q be the weights, MCN the lever: and NC being less than MC, take MD, MA each equal to NC, and $NB = ND$; and let $CN : CM :: P : Q$; P and Q will balance each other.



Let AB be a uniform prism of weight equal to $P + Q$ (Post. 1). Since $MD = CN$, adding CD to both, $MC = DN$; $\therefore AD = 2MD = 2CN$, and $BD = 2DN = 2CM$. Hence

$$AD : BD :: CN : CM.$$

But by supposition, $CN : CM :: P : Q$.

Hence $AD : BD :: P : Q$;

$$\therefore AD + BD : AD :: P + Q : P.$$

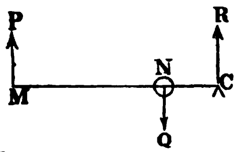
But the prism $AD + BD$ or AB is equal in weight to $P + Q$; $\therefore AD$ is equal in weight to P; and hence BD is equal in weight to Q. Now $AM = CN$, $MC = BN$; hence adding, $AM + MC = CN + BN$, or $AC = BC$: hence the prism AB will balance on the fulcrum C. (Art. 41. Cor. 1.) But (Art. 41.) if the prism AD be attached at its middle point M to the lever MN, and the prism BD at its middle point N, the effect will be the same as before. Therefore in this case also the weights will balance; that is, P at M will balance Q at N.

Cor. 1. The pressure upon the fulcrum is equal to $P + Q$. For the pressure will be the same whether the weights P and Q be collected at M, N, or be extended into the prism AB; and the pressure of the prism AB upon the fulcrum C will be the same as if it were collected at the point C: that is, it will be the weight $P + Q$.

Cor. 2. If P, Q be expressed in numbers, we have $P \cdot CM = Q \cdot CN$; and if this equation be true, the weights P, Q will balance.

Cor. 3. The converse is true, that if the weights balance, they are inversely as the arms. For if not, let $CN : CM :: P : Y$, where Y is different from Q. By the proposition, since $CN : CM :: P : Y$, P and Y balance. But P and Q balance. Now whether Q or Y be the smaller, the one may be made equal to the other by the addition of some weight, as Z; and this being added, the weights which balanced before will balance no longer. Therefore Y is not different from Q; and hence $CM : CN :: P : Q$.

43. Prop. *If two forces acting perpendicularly on a straight lever in opposite directions and on the same side of the fulcrum are inversely as their distances from the fulcrum, they will balance.*



Let MNC be the lever, on which two forces P, Q, act perpendicularly at M, N, in opposite directions, such that $P : Q :: CN : CM$; P, Q will balance.

First, let MNC be a lever on which two forces P, R acting perpendicularly at M, C, on opposite sides of the fulcrum N, balance each other. Then (by Art. 42, Cor. 3,)

$$R : P :: MN : NC, \text{ whence } R + P : P :: MN + NC : NC,$$

$$\text{that is, } R + P : P :: MC : NC,$$

$$\text{but } Q : P :: MC : NC; \ \therefore \ Q = R + P.$$

Now on this lever the pressure on the fulcrum N is $R + P$, in the direction of the forces P and R. (Art. 43, Cor. 1.) Hence the force Q will supply the force which the fulcrum N supplies; and may be put for the fulcrum by Ax. XI. And we may place an immoveable fulcrum at C instead of the force R, by Ax. X., and the equilibrium will still subsist. That is, the forces P, Q will balance on the lever of which the fulcrum is at C, if $P : Q :: CN : CM$.

Cor. 1. The pressure on the fulcrum C is R, which is the difference of $R + P$ and P, that is of Q and P; and is in the direction of the greater, Q.

Cor. 2. The converse of this proposition may be proved nearly as Cor. 3 of Art. 42: that is, if P, Q, acting perpendicu-

larly on the same side of the fulcrum, balance each other, they are inversely as their distances from the fulcrum.

44. When material levers are used, the two forces which have been spoken of, as balancing each other upon the lever, are exemplified by the weight to be raised or the resistance to be overcome, as the one force, and the pressure, weight, or force of any kind employed for the purpose, as the other force. The former of these forces is called the *Weight*, the latter is called the *Power*.

The preceding propositions give the proportion of the Power and Weight in the case of equilibrium, that is, when the weight is not raised, but only supported; or when the resistance is not overcome, but only neutralized. But knowing the power which will produce equilibrium with the weight, we know that any additional force will make the power preponderate. (Ax. II.)

45. Straight levers are divided into three kinds, according to the position of the power and weight.

(1.) The *lever of the first kind* is that in which the power and weight are on opposite sides of the fulcrum, as in Article 42.

We have an example of a lever of this kind, when a bar is used to raise a heavy stone by pressing down one end of the bar with the hand, so as to raise the stone with the other end: the power is the force of the hand, the fulcrum is the obstacle on which the bar rests, the weight is the weight of the stone.

We have an example of a double lever of this kind in a pair of pincers used for holding or cutting; the power is the force of the hand or hands at the handle, the weight is the resistance overcome by the pinching edges of the instrument, the fulcrum is the pin on which the two pieces of the instrument move.

(2.) The *lever of the second kind* is that in which the power and the weight are on the same side of the fulcrum, the weight being the nearer to the fulcrum.

We have an example of a lever of this kind, when a bar is used to raise a heavy stone by raising one end of the bar with the hand, while the other end rests on the ground, and the stone is

raised by an intermediate part of the bar. The fulcrum is the ground, the power is the force exerted by the hand, the weight is the weight of the stone.

We have an example of a double lever of this kind in a pair of nutcrackers. The power is the force of the hand exerted at the handles; the weight is the force with which the nut resists crushing; the fulcrum is the pin which connects the two pieces of the instrument.

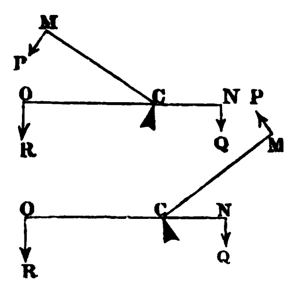
(3.) The *lever of the third kind* is that in which the power and the weight are on the same side of the fulcrum, and the weight is the further from the fulcrum.

In this kind of lever, the power must be greater than the weight, in order to produce equilibrium, by Art. 43. Therefore by the use of such a lever, force is lost. The advantage gained by the lever is, that the force exerted produces its effect at an increased distance from the fulcrum.

We have an example of a lever of this kind in the anatomy of the fore-arm of a man, when he raises a load with it, turning at the elbow. The elbow is the fulcrum; the power is the force of the muscle which, coming from the upper arm, is inserted into the fore-arm near the elbow; the weight is the load raised.

We have an example of a double lever of this kind in a pair of tongs used to hold a coal. The fulcrum is the pin on which the two parts of the instrument turn, the power is the force of the fingers, the weight is the pressure exerted by the coal upon the ends of the tongs.

46. PROP. *If two opposing forces acting perpendicularly at the arms of a bent lever are inversely as the arms, they will balance each other.*



Let MCN be the lever, P, Q the forces which act perpendicularly to CM, CN at M, N.

Produce NC to O, and take $CO = CM$. The lever MCN, being rigid and moveable about C, may be conceived to be part of a rigid material plane MCN moveable about C in its own geometrical plane. (Art. 29.) Hence the point O may be conceived

to be rigidly connected with the lever MCN moveable about C. Let a force R, equal to P, act perpendicularly to the arm CO at O, then (Ax. VI.) the force R produces the same effect as the force P, to turn the lever round C. But the force R will balance the force Q, if

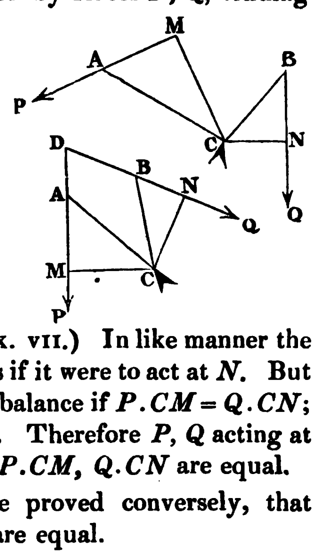
$$R : Q :: CN : CO; \text{ that is, since } P = R, \text{ and } CO = CM,$$
$$\text{if } P : Q :: CN : CM;$$

and therefore P will balance Q, on this supposition.

COR. If $P : Q :: CN : CM$, then $P.CM = Q.CN$: therefore P, Q balance, if $P.CM = Q.CN$, that is, (Def. X.) if the *moments* of the two forces about C are equal.

47. PROP. *On any lever, two opposite forces will balance each other, if their moments round the center of motion are equal.*

Let ACB be any lever acted upon by forces P, Q, tending to turn it in opposite directions, and let CM, CN be perpendiculars upon AP, BQ. Since ACB is a rigid line moveable about C in the plane ACB, it may be considered as a part of a rigid plane ACB moveable about C. (Int. 29.) And since the force P acts at A in the rigid plane ACB, it will produce the same statical effect as if it were to act at any other point, M, of its direction. (Ax. VII.) In like manner the force Q will produce the same effect as if it were to act at N. But the forces P, Q acting at M, N would balance if $P.CM = Q.CN$; that is, if the moments were equal. Therefore P, Q acting at A, B will balance, if their moments $P.CM$, $Q.CN$ are equal.



COR. 1. As before, it may be proved conversely, that if the forces balance, the moments are equal.

COR. 2. If the two forces P, Q act at the same point D, D being rigidly connected with the center C, the proposition is still true. (Art. 24.)

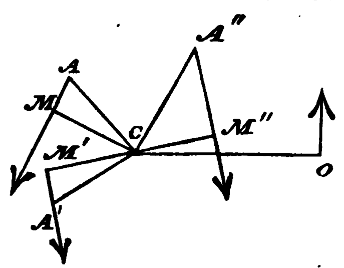
COR. 3. One of the forces will prevail and turn the lever in its own direction, if its moment be greater than that of

the other. Hence the moment of a force *measures* its *effect* to turn the lever round its fulcrum.

Cor. 4. If on any lever ACB a force X act perpendicularly at an arm CO, to turn the lever the same way as P, and if $X.CO = P.CM$, X will produce the same effect as P. For either of the two would balance Q.

48. Prop. *If any number of forces act upon a body, in one plane, tending to turn it round its center in that plane, the moment of the whole is the sum of the moments of the separate forces; opposite forces being reckoned negative.*

By Cor. 3. of last article, the moment of the whole is the measure of the effect of the whole to turn the lever, or body, round its center. Let P, P', P'' be any forces, acting at points A, A', A'', to turn the body round C; let CM, CM', CM'' be perpendiculars on the directions of these forces.



Take any arm CO, and let a force X act perpendicularly at CO, such that $X . CO = P . CM$; also a force X', such that $X' . CO = P' . CM'$: also a force X'', such that $X'' . CO = P'' . CM''$.

Then, by Cor. 4. of last Article, X produces the same effect as P; X', as P'; X'', as P''; therefore the three forces X, X', X'' produce the same effect as the whole of the forces P, P', P''.

If the forces P, P', P'' all tend to turn the body in the same direction, the forces X, X', X'' all tend to turn the body in the same direction, and their whole effect is the effect of their sum, that is, of $X + X' + X''$, acting at CO. And therefore the moment of the whole is

$$= (X + X' + X'')\, CO = X . CO + X' . CO + X'' . CO$$
$$= P . CM + P' . CM' + P'' . CM''$$

by the construction.

If one of the forces, as P'', tend to turn the body in the opposite direction, the corresponding force X'' will also be opposite to the others; and the whole effect will be the effect

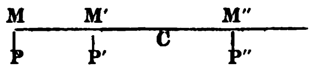
of the excess $X + X' - X''$. In this case the moment of the whole will be

$$P . CM + P' . CM' - P'' . CM''.$$

COR. 1. If any number of forces act upon a body in one plane, tending to turn it round its center in that plane, they will balance when the sum of the moments vanishes: that is, in the case of equilibrium, we have

$$P . CM + P' . CM' - P'' . CM'' = 0.$$

COR. 2. If the forces be parallel forces, acting perpendicularly upon a straight line, and if CM, CM', CM'', be the portions of the lever intercepted between the forces and the fulcrum; we have, in the case of equilibrium,

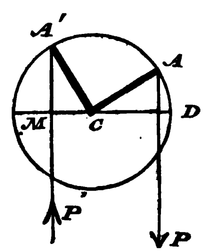


$$P . CM + P' . CM' + P'' . CM'' + \&c. = 0,$$

those values of CM being considered negative which are on the other side of C.

49. *Examples* of several forces acting on a lever.

(1.) A lever has two equal arms at right-angles to each other, which are acted upon by equal forces always acting in the same direction: to find the limits of variation of the force in the course of a revolution.



Let CD be perpendicular to the direction of the forces, $DCA = \theta$, $CA = r$. Therefore the moment is $Pr \cos\theta - Pr \sin\theta$.

Since $\cos 45^0 = \sin 45^0 = \dfrac{1}{\sqrt{2}}$, we may put this in the form,

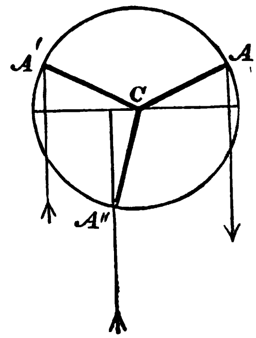
$\sqrt{2} . Pr \{\cos 45^0 \cos\theta - \sin 45^0 \sin\theta\} = \sqrt{2} . Pr \cos(45^0 + \theta)$; which is greatest when $\theta = -45^0$, and is then $\sqrt{2} . Pr$; and vanishes when $\theta = 45^0$.

(2.) When the forces act alternately upwards and downwards, according as the points at which they act move upwards and downwards, (as is the case in the motion produced by a piston,) to find the limits of the variation of force.

In this case, both the forces P, P' always tend to turn the lever the same way. Therefore the moment is $Pr \cos\theta + Pr \sin\theta$, which is $\sqrt{2} . Pr \{\cos 45^0 \cos\theta + \sin 45^0 \sin\theta\} = \sqrt{2} . Pr \cos(\theta - 45^0)$; which is greatest when $\theta = 45^0$, in

which case it is $\sqrt{2} . Pr$; and is least when $\theta = 0$, or $\theta = 90^0$; (for we must not take negative values of $\cos \theta$) in which case it is $\sqrt{2} . Pr \cos 45^0$, or Pr.

Hence, in this case, the amount of the moment alternates between Pr and $\sqrt{2} . Pr$, through every 45^0 of revolution, and never transgresses those limits.



(3.) Let the lever have three equal arms, at angles of 120^0, and let the forces act in the same direction through the whole revolution: to find the variation of the force. The moment $= Pr \cos \theta + Pr \cos (120^0 + \theta) + Pr \cos (240^0 + \theta)$. But $\cos 120^0 = -\frac{1}{2}$, $\sin 120^0 = \frac{\sqrt{3}}{2}$. Hence,

$$\cos (120^0 + \theta) = -\frac{\cos \theta}{2} - \frac{\sqrt{3} \sin \theta}{2}.$$

In like manner, $\cos (240^0 + \theta) = -\frac{\cos \theta}{2} + \frac{\sqrt{3} \sin \theta}{2}$.

Hence the moment $= 0$. In all positions the forces balance.

(4.) In this case let the forces act alternately, as in (2).

If θ be within the limits of 30^0 above and below CD, A' and A'' are in the opposite semi-revolution from A, and the moment is

$$Pr \left\{\cos \theta - \cos (120^0 + \theta) - \cos 240^0 + \theta)\right\} = 2 Pr \cos \theta.$$

But if θ be greater than 30^0 and less than 90^0, A, A'' are in the same semi-revolution, and the moment is

$$Pr \left\{\cos \theta - \cos (120^0 + \theta) + \cos (240^0 + \theta)\right\}$$
$$= Pr (\cos \theta + \sqrt{3} . \sin \theta).$$

In like manner if θ be greater than 30^0 and less than 90^0 below CD, A, A' are in the same semi-revolution. Hence the greatest value of the moment is when $\theta = 0$, in which case it is, by the first formula, $= 2 Pr$. When $\theta = 30^0$, the moment is, by either formula, $\sqrt{3} . Pr$. Take $\theta = 60^0$, and the moment is, by the second formula, $2 Pr$, which is its greatest value. Hence the moment alternates between $\sqrt{3} . Pr$ and $2 Pr$, through each 30^0 of revolution.

50. PROP. *If any number of forces act upon a body moveable about a fixt axis, in any planes perpendicular to the*

axis, the moment of the whole is the sum of the moments of the separate forces; opposite forces being reckoned negative.

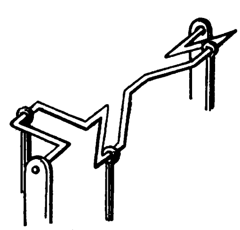
Let the forces be P, P', P''; and let p, p', p'' be the arms, in planes perpendicular to the axis, drawn from the axis perpendicular to the forces respectively. Let any plane be taken perpendicular to the axis, and in this plane let Q be a force equal to P, acting at an arm p. In the same manner let Q', Q'', be forces in this plane equal respectively to P', P'', and acting at arms p', p''. Then by Axiom VIII, the effect of the force Q will be the same as that of P; the effect of Q' will be the same as that of P'; of Q'', as of P'', hence the effect of all the forces P, P', P'' will be the same as the effect of all the forces Q, Q', Q''; therefore the moment of P, P', P'', round the fixt axis will be the same as the moment of Q, Q', Q''; but by Art. 9, the moment of Q, Q', Q'' is $Qp + Q'p' + Q''p''$, that is, $Pp + P'p' + P''p''$. Therefore the moment of P, P', P'', is also $Pp + P'p' + P''p''$.

This proposition is applicable to the Crank, the Wheel and Axle, and the like machines.

51. The *Crank* is a rigid material axis, which in its revolution carries one or more *bends* or *arms*, each arm being fastened to a *rod* by a joint, so that in the course of the revolution the rod has a reciprocating motion.

If there are several arms, they may be in different planes, (the plane of an arm always passing through the axis;) for instance, three arms may be in three planes, making angles of 120^0 with each other.

In this case, if the Crank be moved by the rods, exerting upon it constant forces in alternate directions; if there be but one rod, of which the force is $3P$, the radius of the arm being r, the moment which turns the crank varies between the limits 0 and $3Pr$: but if there be three rods each exerting a force P, the moment varies only between the limits $\sqrt{3}Pr$ and $2Pr$ by Ex. (4.) to last Article. Similarly by Ex. (2.), if there be two arms at right angles to each other, the moment varies between the limits Pr and $\sqrt{2}\,.\,Pr$.



52. The *Wheel and Axle* is a rigid machine, moveable about an axis, consisting of a circular wheel perpendicular to the axis, on which a force acts always in a tangent to the circle; and an axle or cylinder about the axis, on which an opposite force acts in a tangent to the circle made by a section of the cylinder perpendicular to the axis.

In the wheel and axle the moment of a constant force is constant. If P be the force and r the radius of the wheel, it is Pr.

Also, if the resistance at the surface of the axle be constant, its moment is constant. Hence if the power which acts at the circumference of the wheel suffice to overcome the resistance at one instant, it will always suffice.

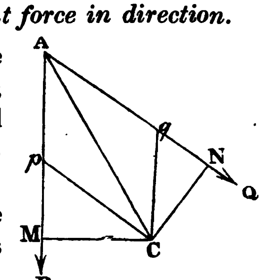
CHAPTER II.

COMPOSITION AND RESOLUTION OF FORCES AT A POINT.

Art. 53. Def. xi. When two forces act at the same point, they produce the same statical effect as a certain single force, acting at that point. This single force is called the *resultant* of the two; they are called its *components*. The two forces produce the single force by being *compounded*, and the single force may be *resolved* into the two.

Def. xii. Straight lines may *represent* forces in direction and magnitude, when they are taken in the direction of the forces and proportional to their magnitude. When forces are so represented, if AB represent any force, BA represents an equal and opposite force. A force represented by any line, as AB, is often called "the force AB."

54. Prop. *If the adjacent sides of a parallelogram represent the component forces in direction and magnitude, the diagonal will represent the resultant force in direction.*



Let Ap, Aq represent in magnitude and direction the forces P, Q, acting at A; complete the parallelogram $ApCq$; and draw AC; draw also CM, CN perpendidular upon Ap, Aq.

The triangles CpM, CqN have right angles at M and N, and the angles MpC, CqN are equal, each being equal

to MAN; therefore the triangles CpM, CqN are equiangular and similar. Therefore $CM : CN :: Cp : Cq$; that is, $CM : CN :: Aq : Ap$. But Ap, Aq represent the forces P, Q in magnitude; therefore $CM : CN :: Q : P$; wherefore $P.CM = Q.CN$. Therefore, by Art. 47, if the forces P, Q act on the plane PAQ, supposed to be moveable about the point C, they will balance each other, producing a pressure on the fulcrum C.

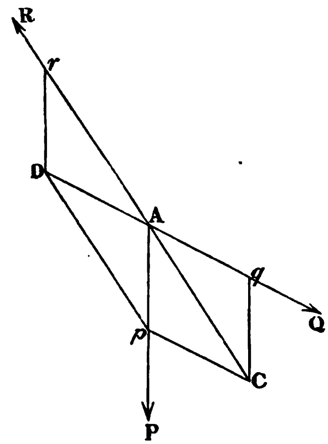
Therefore the single force which produces the same effect as P, Q upon the plane PAQ, will produce a pressure upon the point C, but will not turn the plane about C. But this cannot be the case except the single force act in the line AC; for if it acted in any other direction, a perpendicular might be drawn from C upon the direction, and the force would produce motion, by Axiom II. Therefore the force which produces upon the plane the same effect as P, Q, acts in the direction AC; and is counteracted by the fulcrum at C; that is, (Axiom XI.) by an equal and opposite force acting in CA.

And since this is true when the forces P, Q act on the plane PAQ, moveable about C, it is true when the forces act on any rigid system: for the only effect of the fulcrum C and of the rigidity of the system is to give rise to the reaction in CA; which any other system, in the case of equilibrium, would also do. Hence on whatever system the forces P, Q act, producing equilibrium, they are equivalent to a single force in the direction AC, the diagonal of the parallelogram $ApCq$; that is, the resultant is in the direction AC. Q. E. D.

COR. 1. If a point, acted upon by the two forces Ap, Aq, be kept at rest by a third force, this force must act in the direction CA. For otherwise it would not balance the force in the direction AC, to which the forces Ap, Aq are equivalent.

COR. 2. Hence if three forces act on a point, and keep each other in equilibrium, each of them is in the direction of the diagonal of the parallelogram whose sides represent the other two.

55. PROP. *If the adjacent sides of a parallelogram represent the component forces in direction and magnitude, the diagonal will represent the resultant force in magnitude.*



Let Ap, Aq represent the component forces in direction and magnitude. Complete the parallelogram $ApCq$; then by Art. 54, Cor. 1, the two forces Ap, Aq will be kept in equilibrium by a force in the direction CA. Let Ar represent this force in magnitude. Therefore the three forces Ap, Aq, Ar keep each other in equilibrium. Complete the parallelogram $ApDr$, and draw its diagonal DA. By Art. 54, Cor. 2, the force Aq is in the direction DA, and therefore DAq is a straight line.

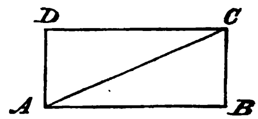
Hence in the triangles CAq, DAr, the vertical angles CAq, DAr are equal, and Cq, Dr are parallel to each other, because Cq and Dr are both parallel to Ap; also Cr meets them, therefore the angle qCA is equal to the alternate angle DrA. Therefore the triangles CAq, DAr are equiangular. Also Cq and Dr are equal, for each is equal to Ap, being opposite sides of parallelograms pq, pr. Therefore the other sides of the triangle CAq, DAr are equal; therefore CA is equal to Ar. But the force which keeps in equilibrium Ap, Aq, is Ar acting in the opposite direction. Therefore AC, which is equal to Ar, represents in magnitude the force which produces the same effect as Ap, Aq; that is, AC represents the resultant of Ap, Aq. Q.E.D.

Cor. If the components be represented in magnitude and direction by the sides of a parallelogram, the resultant is represented in magnitude and direction by the diagonal of the parallelogram.

56. Prop. *Any force represented in magnitude and direction by one side of a triangle, is equivalent to two forces represented by the two other sides of the triangle.*

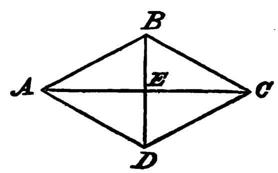
Let a force be represented in magnitude and direction by the straight line AC; this force is equivalent to two forces AB, BC.

Complete the parallelogram $ABCD$. Two forces AB, AD are equivalent to a force AC, by Cor. to Art. 54. But the force which is represented by AD may be represented by BC. (Def. XII.) Therefore the two forces AB, BC are equivalent to AC, and AC is equivalent to AB, BC: that is, AC may be resolved into AB, BC.



Cor. 1. If ABC be a right angle, R the force AC, θ the angle CAB, R is equivalent to two forces $R \cos \theta$ and $R \sin \theta$ in rectangular directions AB, AD. For $AB = AC \cos \theta$, and $BC = AC \sin \theta$.

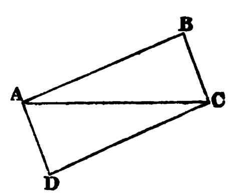
Cor. 2. If AB, AD be equal, AC bisects the angle BAD; and if $AB = P$, $AC = R$, and the angle $BAD = A$, $R = 2P \cos \frac{1}{2} A$. For the two diagonals AC, BD, of the rhombus $ABCD$ bisect each other at right angles. Hence, $AC = 2AE$, and $AE = AB \cos BAE = AB \cos \frac{1}{2} A$, whence the formula is manifest.



Cor. 3. If a force be resolved in two given directions, the components thus obtained will be the same, at whatever point of the force the resolution be supposed to take place.

57. Prop. *If three forces, represented in magnitude and direction by the sides of a triangle taken in order, act on a point, they will keep it in equilibrium.*

Let three forces, represented in magnitude and direction by the three lines AB, BC, CA, act on the point A, they will keep it in equilibrium.



The forces AB, BC are equivalent to a force AC, (Art. 55); therefore the forces AB, BC, CA will produce the same effect as AC, CA; that is, they will keep the point A in equilibrium.

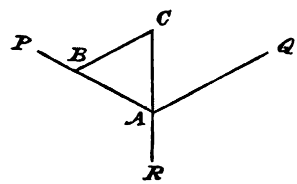
Cor. 1. If three forces which keep a point in equilibrium be in the direction of three lines forming a triangle, they are proportional to those lines.

Cor. 2. If three forces act in three directions, and if lines perpendicular to these three directions be drawn, forming a triangle, the sides of this triangle will be proportional to the forces.

For it is easily shewn that three lines being drawn perpendicular to three other lines, the triangle formed by the first three lines is equiangular with the triangle formed by the other three lines: and therefore its sides are in the same proportions as the sides of the other; which, by Cor. 1, are as the forces.

58. *If three forces keep a point in equilibrium, they are each as the sine of the angle made by the other two.*

Let P, Q, R be three forces which keep a point A in equilibrium. Draw BC parallel to AQ meeting the directions of the other two forces in B, C; then the sides of the triangle ABC will be proportional to the forces because they are in the direction of the forces (Art. 36, Cor. 1.) Therefore



$$P : Q :: AB : BC :: \sin ACB : \sin BAC :: \sin CAQ : \sin CAP;$$

$$\therefore P : Q :: \sin QAR : \sin PAR.$$

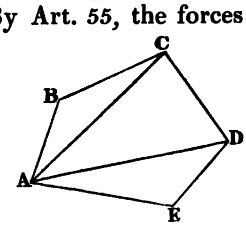
In like manner we may prove

$$P : R :: \sin RAQ : \sin PAQ,$$

whence, $Q : R :: \sin RAP : \sin QAP$.

59. Prop. *If any number of forces, represented in magnitude and direction by the sides of a polygon taken in order, act on a point, they will keep it in equilibrium.*

Let forces AB, BC, CD, DE, EA act upon a point A; they will keep it in equilibrium. By Art. 55, the forces AB, BC are equivalent to the force AC; therefore the forces AB, BC, CD are equivalent to the force AC, CD; that is, by the same Article, to a force AD. Therefore again, the forces AB, BC, CD, DE are equivalent to the forces AD, DE; that is, again by the same Article, to a force AE. Therefore, finally,

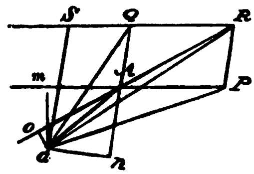


the forces AB, BC, CD, DE, EA are equivalent to forces AE, EA, and therefore will keep the point A in equilibrium.

Cor. The reasoning is applicable, and therefore the result is true, when the sides of the polygon are in different planes.

60. Prop. *The moment of the resultant of two forces acting in one plane about any point is equal to the sum or difference of the moments of the components; according as the forces tend to turn the system the same or different ways.*

Let AP, AQ represent two forces, P, Q; AR, the diagonal of the parallelogram $APRQ$, their resultant R. Let a be any point, am, an, ao perpendiculars upon PA, QA, RA: then



$$R.ao = P.am - Q.an.$$

Draw aS parallel to AQ meeting QR in S. Then

$$\text{triangle } ARP = \tfrac{1}{2}\,\text{par}^{\text{m}}\, PQ = \tfrac{1}{2}\,\text{par}^{\text{m}}\, PS - \tfrac{1}{2}\,\text{par}^{\text{m}}\, AS$$
$$= \text{tri. } aPR - \text{tri. } aAQ$$
$$= \text{quadril. } aPRA - \text{tri. } aRA - \text{tri. } aAQ,$$

Hence, tri. aRA = quadril. $aPRA$ − tri. ARP − tri. aAQ

or, tri. aRA = tri. aAP − tri. aAQ;

Hence, doubling both sides and putting $\frac{1}{2}$ base × perpen. for the triangles,

$$AR.ao = AP.am - AQ.an,$$

that is, $R.ao = P.am - Q.an.$

And in the same manner, if a be so situate that P and Q tend to move the system the same way, we may prove that

$$R.ao = P.am + Q.an.$$

Cor. 1. If a be the origin of rectangular co-ordinates; P, Q parallel respectively to those co-ordinates; x, y the co-ordinates of the point A, $ao = r$; we have

$$Rr = Py - Qx.$$

Cor. 2. If a force, not in the plane of motion, act to turn a body about any axis, and if x, y be the co-ordinates of the point of application of the force in a plane perpendicular to the axis; X, Y the components of the force parallel to the co-ordinates; Rr the resulting moment; we have

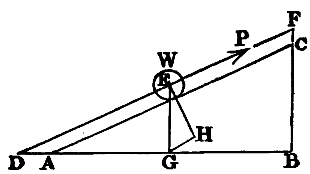
$$Rr = Yx - Xy.$$

For the component of the force parallel to the axis (which we may call Z) is destroyed by the forces which constrain the body in that direction.

Cor. 3. If any number of forces act upon a body to move it round an axis; and if X, Y; X', Y'; &c. be the components in the directions of rectangular co-ordinates in a plane perpendicular to the axis; we have

$$Rr = Yx - Xy + Y'x' - X'y' + \&c.$$

61. Prop. *On an inclined plane, the force which, acting parallel to the plane, will support a weight* W, *is* $\frac{WH}{L}$; *where* H *is the height of the inclined plane, and* L *its length.*

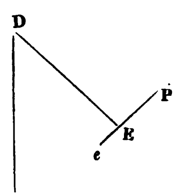


Let AC be an inclined plane, of which AC is the length, L, and BC the height, H, AB being horizontal. And let the body W be supported by a force P, acting in the direction EF parallel to AC.

The force which W exerts in virtue of its weight is vertical, and may be represented by a vertical line EG. Draw EH perpendicular and GH parallel to the plane. The force EG is equivalent to EH, HG, of which EH is destroyed by the reaction of the plane. (Ax. xii.) Hence the weight will be sustained if W is acted upon by a force at E, represented by GH: that is, by a force P, parallel to GH or AC, and such that $P : W :: GH : GE$. But angle $GEH = CAB$; hence the triangles GEH and CAB are similar; and $GH : GE :: CB : CA :: H : L$. Therefore $P : W :: H : L$, and $P = \frac{WH}{L}$.

Cor. 1. If the angle $CAB = \alpha$, $P = W \sin \alpha$.

Cor. 2. If the body rest on a curved surface instead of an nclined plane, and if AC be parallel to the tangent at the point where the body touches the surface, we have still $P = W \sin \alpha$. For the reaction is still perpendicular to AC.

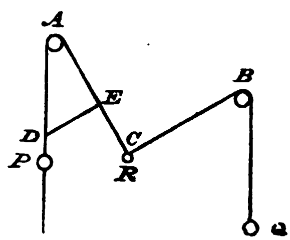


Cor. 3. If the body W, instead of being sustained on a surface, be sustained by a string DE,

fastened to a fixed point D; and if a be the angle which DE makes with the vertical line, P, the force which, acting perpendicular to DE, supports W, we have still $P = W \sin a$. For by the string, the body is compelled to describe an arc Ee (perpendicular to DE) if it move at all; and is in the same condition, with respect to the forces that support it, as it is when sustained upon a surface Ee, as in the last corollary. The action of the string supplies the place of the reaction of the surface. Also a, the angle which DE makes with the vertical, is equal to the angle which the surface Ee, perpendicular to DE, makes with the horizon.

62. *Examples* of the resolution and composition of forces.

Ex. (1.) A given weight, R, is supported at a knot by two given weights P, Q, fastened to it by strings which pass over given fixt pullies: to find the position of equilibrium.

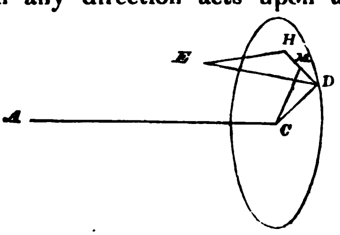


Let A, B be the pullies: C the knot.

The tensions of the strings CA, CB are P, Q; the point A being in the string CA, take any vertical line AD; and draw DE parallel to CB. Then the sides of the triangle ADE represent the forces R, Q, P. Hence if, AD being as R, we construct a triangle ADE, of which the sides AE, DE are as P, Q, this triangle gives the direction of AC; and if we draw BC parallel to ED, we have the point C.

The angles CAP, CBQ are then given by the formulæ for finding the angles of a triangle when the three sides are given.

63. *Ex.* (2.) A force in any direction acts upon a rigid body moveable about a fixt axis: to find the moment about the axis.

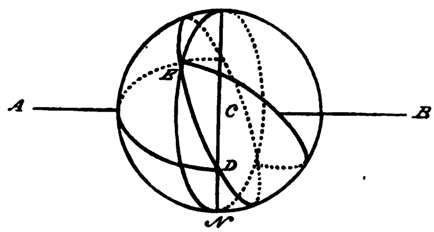


Let AC be the axis, ED the direction of the force, CDH a plane perpendicular to the axis, EH a perpendicular upon this plane, CM a perpendicular upon DH.

If ED represent the force, ED is equivalent to forces EH, HD, of which EH is destroyed by the forces which con-

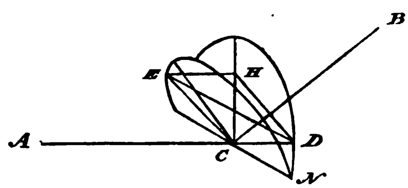
strain the axis longitudinally; and HD acts to turn the body round the axis with a moment $HD \cdot CM$. Hence the moment of the force is $HD \cdot CM = ED \cos EDH \cdot CD \sin CDH = R \cdot CD \cos EDH \cdot \sin CDH$, if R be the magnitude of the force.

64. *Ex.* (3.) In a *Hook's Joint*, the moment is constant about the first axis: to find the variation of the moment about the second axis.



Hook's Joint consists of two rods, each moveable about its axis; which axes meet. The rods are connected with each other, being fastened by hinges to the same rigid body, (ring, cross, or sphere) in such a manner that the axis of the hinge of each rod is perpendicular to the axis of the rod, and the two axes of the hinges perpendicular to each other in the same plane.

Let AC, BC be the rods; CD, CE the axes of the hinges; let the plane NCD, perpendicular to AC, and NCE, perpendicular to BC, be the planes in which D moves round AC, and E round BC by the respective motions of the rods round their own axes. Let P be the force, and $P \cdot CD$ the moment, about the axis AC; and let Q be the force, and $Q \cdot CE$ the moment, about the axis BC.



The two rods act upon each other by means of a rigid body connected with them respectively at the points D, E; hence, the mutual action is in the straight line DE: let this mutual action be R.

If EH be drawn perpendicular to the plane NCD, and HD be joined, we shall have by the last example, since the moments of P and R must balance about the axis AC,

$$P \cdot CD = R \cdot CD \cdot \cos EDH \cdot \sin CDH;$$

$$P = R \cos EDH \cdot \sin CDH.$$

$$\text{Now } \sin EDH = \frac{EH}{ED} = \frac{CE \sin ECH}{ED};$$

but CE and CH are both perpendicular to CD; hence the angle ECH measures the inclination of the plane CDE to the plane CDN; let this be called D.

Also, let $CE = CD$; and, since the angle DCE is a right angle, $ED = \sqrt{2} \cdot CE$.

$$\text{Hence, } \sin EDH = \frac{\sin D}{\sqrt{2}};$$

$$\text{whence } \cos EDH = \sqrt{\left(1 - \frac{\sin^2 D}{2}\right)} = \frac{\sqrt{(1 + \cos^2 D)}}{\sqrt{2}}.$$

$$\text{Also, } \tan CDH = \frac{CH}{CD} = \frac{CE \cos D}{CD} = \cos D;$$

$$\text{hence, } \sin CDH = \frac{\cos D}{\sqrt{(1 + \cos^2 D)}},$$

$$\text{hence, } P = R \cdot \cos EDH \cdot \sin CDH = R \frac{\cos D}{\sqrt{2}}.$$

In like manner if E be the angle of inclination of the plane CED to the plane CEN, we shall find

$$Q = R \frac{\cos E}{\sqrt{2}}.$$

$$\text{Hence, } \frac{Q}{P} = \frac{\cos E}{\cos D}.$$

But if the planes NCD, NCE, DCE cut a sphere described with centre C, they will form a quadrantal triangle, the side DE being a quadrant: and two angles of this triangle will be D and E: and the third angle (N) will be the angle of inclination of the planes DCN, ECN. And by the rules for quadrantal triangles, if ND be x,

$$\text{rad} \cdot \cos x = \tan D \cdot \text{cotan } N; \text{ and } \text{rad} \cdot \cos N = \cos D \cdot \cos E.$$

$$\text{Hence, } \tan D = \tan N \cos x; \quad \frac{1}{\cos^2 D} = 1 + \tan^2 N \cos^2 x,$$

$$\text{whence, } \frac{\cos E}{\cos D} = \frac{\cos D \,.\, \cos E}{\cos^2 D} = \cos N(1 + \tan^2 N \cos^2 x)$$

$$= \frac{1 - \sin^2 N \sin^2 x}{\cos N}.$$

$$\text{Hence, } \frac{Q}{P} = \frac{1 - \sin^2 N \sin^2 x}{\cos N}.$$

$$\text{When } x = 0, \quad Q = \frac{P}{\cos N};$$

$$\text{when } x = 90^0, \quad Q = P \cos N,$$

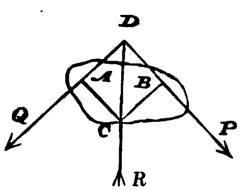
these are the limits of Q's variation.

CHAPTER III.

THE EQUILIBRIUM OF A RIGID BODY.

65. Prop. *When a rigid body is kept in equilibrium about a fixt point by two forces, the pressure upon the fixt point is the same as if the two forces were transferred to the point retaining their magnitude and direction.*

Let P, Q be two forces acting in directions BP, AQ, to turn a body about a fixt point C, and producing equilibrium. Draw CA, CB parallel to P, Q respectively. The forces P, Q may be supposed to act at the point D in which their directions meet, and their resultant must pass through the point C, because they produce equilibrium (Art. 53). Hence (Art. 54,) the forces may be represented by DA, AC, and their resultant by DC; and DC is the pressure upon C. Hence the pressure upon the fixt point C is the same as if the forces P, Q, acted in the lines AC, BC.



It is not requisite that the point D should be within the body.

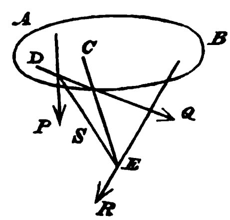
66. Prop. *When three forces keep a rigid body in equilibrium, each of them passes through the intersection of the other two, and is equal and opposite to the resultant of the other two.*

Let P, Q, R be three forces (see the figure in last Article) which keep a rigid body AB in equilibrium. The forces P, Q produce the same effect as if they acted at D, and therefore cannot be kept in equilibrium by a force which does not pass through D. But R balances the forces P, Q; hence the direction of R passes through D.

Also (Ax. XI.) instead of the force R, we may suppose a fulcrum at C, exerting the same force as R. But since there is equilibrium, by last Prop. the force of the fulcrum C is equal and opposite to the resultant of P, Q; therefore the force R is equal and opposite to the resultant of P, Q.

This Proposition will be exemplified in treating of the equilibrium of structure.

67. PROP. *When any forces in one plane keep a rigid body in equilibrium about a fixt point, the pressure on the fixt point is the same as if the forces were all transferred thither, retaining their magnitude and direction.*



Let P, Q, R, be any forces which keep a rigid body AB in equilibrium about C: P, Q produce the same effect as if they acted at their point of concourse D; that is, they produce a pressure S at D in the direction DE, of which pressure S, the components parallel to DP, DQ are P, Q. But this force S may be supposed to act at E, and is equivalent to a pressure S at E in the direction DE, of which pressure the components parallel to DP, DQ are P, Q (Art. 56, Cor. 3). Let this pressure S be compounded with the force R, which may also be supposed to act at E, a point of its direction; and let the resultant of S, R be a force in the direction CE. This force may be supposed to act at C in the direction CE; and its components in the directions parallel to DE, ER are S, R. But the components of S in the directions parallel to DP, DQ, are P, Q; hence the force which acts at C has for its components parallel to P, Q, R, the forces P, Q, R. And by continuing this construction, if there be more forces, all the forces will be equivalent to a single force acting in the last line CE, thus found, and having for its components parallel to P, Q, R,

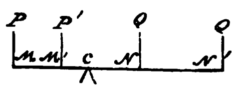
the forces P, Q, R. But the last line thus found must pass through the fixt point C, otherwise the forces could not produce equilibrium about C. Therefore on the fixt point C there is a pressure equivalent to P, Q, R, acting in their proper directions.

Cor. 1. In the same manner as in Art. 23, it appears that if any number of forces in one plane act upon a body, and keep it in equilibrium, each must coincide in direction with the resultant of all the others, and must be equal and opposite to that resultant.

Cor. 2. It is not requisite that the points D, E, C should fall within the body.

68. Prop. *When any parallel forces in one plane keep a rigid body in equilibrium about a fixt point, the pressure upon the fixt point is equal to the sum of the forces.*

This might be considered as included in the last proposition, and proved in the demonstration of that. But it may be thus proved independently. Let any forces P, P' on one side of a fixt point C, and Q, Q' on the other, balance each other. Then, by the property of the lever, (Art. 48),



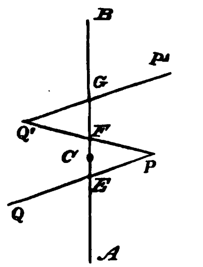
$$P . CM + P' . CM' - Q . CN - Q' . CN' = 0.$$

Let $P . CM$, a moment on one side, be greater than $Q . CN$, a moment on the other; and let $u . CM = Q . CN$ and $u + p = P$. Put $u + p$ in the place of P: then u and Q balance each other, and therefore the remaining forces p, P', Q' also balance. In the same manner let $v' . CN' = P' . CM'$, and $v' + q' = Q'$: then v' and P' balance; and since p, P', Q', that is, p, P', $v' + q'$, balance, of which v' and P, balance, therefore also p and q' balance. Hence the forces P, P', Q, Q' are separated into pairs Q and u, P' and v', p and q', each of which pairs balance on C. And hence the pressure on C is (Art. 43)

$$Q + u + P' + v' + p + q' = p + u + P' + Q + q' + v'$$
$$= P + P' + Q + Q'.$$

And in the same manner this proposition may be proved for any other forces.

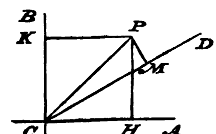
COR. If the forces be not in the same plane, let them meet in P, P', Q, Q' the plane perpendicular to them, and balance upon the line AB in that plane. Separate the forces into pairs as above, and join Q and P, P and Q', Q' and P', meeting AB in E, F, G. Then by Art. 48 the forces Q, u produce a pressure at E equal to $Q+u$; the forces P', v', produce at G a pressure $P'+v'$; and the forces p, q' a pressure $p+q'$ at F. And these pressures at E, F, G balance upon C, the point on which all the forces balance; and therefore the pressure on C is $Q+u+P'+v'+p+q'$ by the Proposition; that is, it is



$$P+P'+Q+Q'.$$

69. PROP. *When any parallel forces, acting on a rigid body, balance about each of two straight lines, perpendicular to one another in the plane perpendicular to the forces, they will balance about any straight line in that plane, drawn through the intersection of the other two.*

Let CA, CB be the two straight lines perpendicular to each other in the plane perpendicular to the forces, CD any other line; P the point where one of the forces meets this plane; PH, PK perpendiculars on CA, CB; PM perpendicular on CD.



Let $ACP=\theta$, and $ACD=a$, $CP=r$. Hence, $PCD=\theta-a$. Then, $PM=CP.\sin PCM=r\sin(\theta-a)=r\sin\theta\cos a-r\cos\theta\sin a$. But, $r\sin\theta=PH$, $r\cos\theta=CH=PK$. Hence, $PM=PH\cos a-PK\sin a$.

Let P', H', K', M' represent corresponding points for another force P'; then we have $P'M'=P'H'\cos a-P'K'\sin a$. And similarly for other forces. Hence, $P.PM+P'.P'M'+$ &c. $=(P.PH+P'.P'H'+\text{&c.})\cos a-(P.PK+P'.P'K'+\text{&c.})\sin a$. And when all the forces are taken, and the perpendiculars PH which are on the other side of CA are considered as negative, and also the perpendiculars PK which are on the other side of CB, since the forces balance about CA and CB, we have by Art. 48

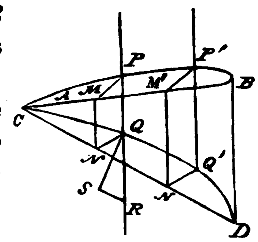
$$P.PH+P'.P'H'+\text{&c.}=0,\quad P.PK+P'.P'K'+\text{&c.}=0.$$

$$\text{Hence, } P.PM + P'.P'M' + \&c. = 0,$$

therefore, P, P', &c. balance about CD.

70. Prop. *If any parallel forces acting on a rigid body balance about a straight line lying in a plane perpendicular to the forces, and if a plane be drawn through this straight line parallel to the forces, the forces will balance about any other fixt straight line lying in the plane so drawn, and making a finite angle with the forces.*

Let any parallel forces P, P', &c. balance about a line AB, lying in a plane perpendicular to them; and let CD be any line in the plane which passes through AB and is parallel to the forces: the forces will balance about CD.



The forces P, P', &c. which balance about the line AB may be supposed to act at the points where they meet the plane perpendicular to the forces in which AB lies; and PM, $P'M'$, &c. being the perpendiculars drawn from these points respectively upon AB, we have $P.PM + P'.P'M' + \&c. = 0$; because of the equilibrium about AB.

Let CQD be a plane perpendicular to the plane BCD; and let the force P meet this plane in Q; and let a plane NQS, perpendicular to the line CD, be drawn, and a plane RQS parallel to BCD. The line CD is perpendicular to the plane NQS; therefore the plane BCD is perpendicular to the plane NQS, and the plane NQS to the plane BCD. And the plane CQD is also perpendicular to the plane BCD; hence the line QN, the intersection of CQD and NQS, is perpendicular to BCD; and hence QN is perpendicular to the line CD. Also QN is perpendicular to the plane RQS (which is parallel to BCD,) and therefore to the line QS. Let SR parallel to CD meet PQ in R; hence SR is perpendicular to the plane NQS, and QSR is a right angle. Also the angle QRS is the same for all the points Q, being the angle which the line parallel to CD makes with the forces; let this angle be α.

The force P may be supposed to act at Q (Ax. VII.); and being represented by QR, may be resolved into QS, SR, of which SR acts parallel to the axis CD, and produces no effect to

turn the body round that axis. The force QS acts to turn the body round the axis CD; and since $QS = QR.\sin QRS$, this force $= P \sin \alpha$. And the same for each of the other forces. Now $P.PM + P'.P'M' + \&c. = 0$. And multiplying by $\sin \alpha$,

$$P \sin \alpha . PM + P' \sin \alpha . P'M' + \&c. = 0.$$

Or, since QN is equal to PM,

$$P \sin \alpha . QN + P' \sin \alpha . Q'N' + \&c. = 0.$$

Hence the forces balance about the line CD.

CHAPTER IV.

ANALYTICAL FORMULÆ OF STATICS.

71. When any forces P, P', &c. in one plane act upon a point; to find the resultant.

If each force be resolved in the direction of two rectangular co-ordinates x, y; and if X, Y be the components of P; X', Y' the components of P'; and so on; then, R being the resultant of all the forces, and θ the angle which it makes with the axis of x; we shall have

$$R \cos \theta = X + X' + \&c.: \quad R \sin \theta = Y + Y' + \&c.$$

whence R and θ are known.

For $R \cos \theta$ is the component of R in the direction of x, and $R \sin \theta$ is the component of R in the direction of y: and these must be the sums of the components of the separate forces in these directions.

72. On the same suppositions, to find the equations of equilibrium of forces acting at a point.

The equations are

$$X + X' + \&c. = 0, \quad Y + Y' + \&c. = 0;$$

and these equations are necessary and sufficient, as conditions of equilibrium.

They are necessary, for the equilibrium cannot subsist, if the point be acted upon by any resulting uncounteracted force; hence $R = 0$, and therefore $R \cos \theta = 0$, and $R \sin \theta = 0$, which give the above conditions.

They are sufficient, for if the body be acted upon by no force, or by forces which destroy each other, it will be in equilibrium.

73. If any forces P, P', &c. act upon a point in any directions, to find their resultant.

Let the forces be resolved in the directions of three rectangular co-ordinates x, y, z; and let X, Y, Z be the components of P; X', Y', Z' the components of P'; and so on. Also let R be the resultant, and θ, η, ζ the angles which it makes respectively with x, y, z: then

$$R\cos\theta = X + X' + \&c.;\quad R\cos\eta = Y + Y' + \&c.;$$
$$R\cos\zeta = Z + Z' + \&c.$$

Also, $\cos^2\theta + \cos^2\eta + \cos^2\zeta = 1$.

These equations determine R, θ, η, ζ.

74. If any forces act upon a point in any directions, and be resolved in the directions of three rectangular co-ordinates, x, y, z; to find the equations of equilibrium.

The necessary and sufficient conditions of equilibrium are,

$$X + X' + \&c. = 0;\quad Y + Y' + \&c. = 0;\quad Z + Z' + \&c. = 0.$$

The same reasoning applies in this case as in Art. 72.

75. If any forces P, P', &c. in one plane, act upon a rigid body; to find their resultant.

Let P be one of the forces, x, y the co-ordinates of its point of application, X, Y its components: and let P', x', y', X', Y', be the corresponding quantities for another force; and so on. Also let R be the resultant; s, t, the co-ordinates of its point of application; θ the angle which it makes with x. Then

$$R\cos\theta = X + X' + \&c.;\quad R\sin\theta = Y + Y' + \&c.$$
$$R\cos\theta\,.\,t - R\sin\theta\,.\,s = Xy - Yx + X'y' - Y'x' + \&c.$$

the two former equations are true, because the components and the resultant, being resolved in any given directions, must give identical results. The latter equation is true, because the moment of the resultant with regard to any given point (and therefore with regard to the origin of co-ordinates) must be equal to the moment of the components (Art. 60).

The two former equations determine R and θ.

The third equation does not enable us to determine s and t, but only gives an equation between them; and this is the equation to the straight line in which R acts.

76. **If any parallel forces P, P', &c. act upon a rigid body, to find their resultant.**

Let a plane be taken perpendicular to the forces, and let x, y be the co-ordinates of the point where P meets this plane; x', y' the co-ordinates for P'; and so on. Also let s, t be the co-ordinates of the point where the resultant R meets this plane. Then R will be parallel to P, P', &c., and

$$R = P + P' + \&c.;$$

$$Rs = Px + P'x' + \&c.; \quad Rt = Py + P'y' + \&c.$$

Hence, R, s, t.

This appears from the property of the lever; for the parallel forces may be considered as acting upon a lever to turn it about an axis passing through the origin parallel to x, or to y. And the resultant must produce the same effect, in these ways, as the components.

77. **If any forces P, P', &c. act upon a rigid body, to find their resultant.**

Let X, Y, Z be the components, x, y, z the co-ordinates of the point of application of P; X', Y', Z', x', y', z' the same for P'; and so on. Also let R be the resultant, θ, η, ζ the angles which it makes with the axes x, y, z; s, t, u the co-ordinates of its point of application. Then we have

$$R\cos\theta = X + X' + \&c.; \quad R\cos\eta = Y + Y' + \&c.;$$

$$R\cos\zeta = Z + Z' + \&c.$$

$$R\cos\theta \,.\, t - R\cos\eta \,.\, s = Xy - Yx + X'y' - Y'x' + \&c.$$

$$R\cos\eta \,.\, u - R\cos\zeta \,.\, t = Yz - Zy + Y'z' - Z'y' + \&c.$$

$$R\cos\zeta \,.\, s - R\cos\theta \,.\, u = Zx - Xz + Z'x' - X'z' + \&c.$$

Also $\cos^2\theta + \cos^2\eta + \cos^2\zeta = 1$.

The three former equations express that the components and the resultant being resolved in any given directions, must give identical results.

The fourth equation expresses that the moment of the resultant round the axis of z is equal to the moment of the components; which it must be, by Art. 60, Cor. 2.

The fifth and sixth express the same with regard to the axes x and y.

For the forces parallel to z constrain the body in the direction of z, and make the condition of equilibrium the same as if the body were moveable about a *fixt* axis parallel to z. And the same may be said with respect to the axes x and y.

We have, above, seven equations, and seven unknown quantities, R, θ, η, ζ, s, t, u. But there are only six equations really independent; and hence the resultant is not possible except under certain conditions. Two of the equations in s, t, u are the equations to the straight line in which R acts, if a single resultant of the forces is possible. In this case the third equation in s, t, u, is involved in the other two, as will appear hereafter.

78. If any forces P, P', &c., as before, act upon a rigid body, to find two forces to which they are equivalent.

The force P, which has for its components X, Y, Z, and for the co-ordinates of its point of application, x, y, z, may be supposed to act at any point of its direction. Let it act at the point where it meets the plane xy. Since its components parallel to y, z are Y, Z, it is evident that if it be continued backwards to the plane xy, it will meet it in a point of which the co-ordinate y is less by $\frac{Yz}{Z}$ than that of the point of application where the co-ordinate is z. Hence the co-ordinate, parallel to y, of the point where P meets the plane xy, is $y - \frac{Yz}{Z}$. In like manner, $x - \frac{Xz}{Z}$ is the co-ordinate of this point parallel to x. And so of the rest.

The forces P, P', &c. acting at these points, are resolved into Z, Z', &c. and X, X', &c. Y, Y', &c. Let the resultant of the latter forces be a force (in the plane x, y) of which the components parallel to x and y are S and T, and the co-ordinates of its points of application, s, t.

Also we shall have S, T, s, t, by Art. 74. But instead of $Xy - Yx$ we must put

$$X\left(y - \frac{Yz}{Z}\right) - Y\left(x - \frac{Xz}{Z}\right), \text{ which is still } = Xy - Yx.$$

Hence we have,

$$St - Ts = Xy - Yx + X'y' - Y'x' + \&\text{c.} = L, \text{ suppose.}$$

And, $S = X + X' + \&c.$, $T = Y + Y' + \&c.$

Hence, S, T and $St - Ts$ are found.

Also let U be the resultant of the parallel forces, Z, Z', &c.; v, w, the co-ordinates parallel to x, y of the point where this resultant meets the plane xy. Therefore by Art. 74,

$$Uv = Z\left(x - \frac{Xz}{Z}\right) + Z'\left(x' - \frac{X'z'}{Z'}\right) + \&c.$$

or, $Uv = Zx - Xz + Z'x' - X'z' + \&c. = M$, suppose.

Similarly,

$$Uw = Zy - Yz + Z'y' - Y'z' + \&c. = -N, \text{ suppose}$$

And $U = Z + Z' + \&c.$

Hence U, v, w, are known.

By what precedes, we obtain, as the resultant of P, P', &c. two forces; one, U, perpendicular to the plane xy; the other, the resultant of S, T, in the plane xy.

79. To find under what condition the forces P, P', &c. acting upon a rigid body, have a single resultant.

The forces P, P', &c. are equivalent to U, and to the resultant of S, T, found as above.

Now if these two forces meet each other, they may be compounded into a single force at the point of concourse: but if the two resulting forces do not meet each other, the forces P, P', &c. cannot be reduced to a single resultant. But the forces will meet each other, if at all, in the plane xy; that is, they will meet, if U meet this plane in a point which is in the line of action of the force compounded of S, T; that is, in a point of the line given by the equation

$$St - Ts = L.$$

Hence we may put, for s and t, the values of v and w, which are, $v = \frac{M}{U}$, $w = -\frac{N}{U}$; and we have

$$-NS - MT = LU; \text{ or, } LU + MT + NS = 0,$$

an equation of condition requisite in order that the forces P, P', &c. may have a single resultant.

Cor. 1. In the case in which there is a single resultant, R, it is plain that S, T, U are its three components in the directions x, y, z respectively; that is, in the notation of Art. 76, they are $R\cos\theta$, $R\cos\eta$, $R\cos\zeta$. Hence the three equations of that Article, in s, t, u, become

$$St - Ts = L, \quad Tu - Ut = N, \quad Us - Su = M.$$

But if we multiply the first by U, and the second by S, and add, we have $STu - TUs = LU + NS$, and this is $= -MT$, in virtue of the equation of condition. Hence dividing by $-T$,

$$Us - Su = M,$$

that is, the third equation is involved in the other two, as was before asserted (Art. 76).

Cor. 2. Hence the solution of this article agrees with the solution of Art. 76.

80. To find the necessary and sufficient conditions of equilibrium of any forces acting upon any rigid body.

The conditions are, retaining the former notation,

$$X + X' + \&c. = 0, \quad Y + Y' + \&c. = 0, \quad Z + Z' + \&c. = 0,$$
$$Xy - Yx + X'y' - Y'x' + \&c. = 0,$$
$$Zy - Yz + Z'y' - Y'z' + \&c. = 0,$$
$$Xz - Zx + X'z' - Z'x' + \&c. = 0.$$

These conditions of equilibrium are necessary and sufficient. They are *necessary*: for if the three first be not satisfied, there will be a resulting force unbalanced, and therefore there cannot be equilibrium. And if the three last (the moments) be not satisfied, there will be motion about some axis. Thus, suppose the two former of the three conditions of moments to be satisfied, and the last unsatisfied; then there can be no motion about the axis of z by the first of the three; nor about the axis of x, by the second; nor about any axis in the plane xz, passing through the origin, by Art. 69; nor about any axis passing through the axis y and inclined to the plane xz, by Art. 70. But there may still be motion about the axis y, perpendicular to the plane xz; and there will be such motion, except the third condition of moments is also satisfied.

Also these equations are *sufficient:* for if the three conditions of moments are satisfied, there can be no motion about any axis passing through any of the co-ordinate axes x, y, z; for the reasons just given: that is, no motion about any axis at a finite distance from the points where the forces act: for the ordinates x, x', &c. are supposed to be finite. Also if the equations be satisfied, there can be no motion about an axis at an infinite distance; for this is equivalent to a motion of translation of the body parallel to itself; which does not take place, because the resultant of the forces vanishes. Therefore there is no motion about any axis, at a finite or infinite distance. But all motion is motion about some axis. Therefore there is no motion produced, and the equations are *sufficient* for equilibrium.

There is one case of forces acting upon a rigid body which offers some remarkable results, namely, where the forces are not reducible to a single force, but are reducible to two equal forces in opposite directions, not in the same straight line. A combination of two such forces is called a *Statical Couple.* For the properties of statical couples, see the *Mechanics of Engineering*, Chap. III.

CHAPTER V.

THE CENTER OF GRAVITY.

81. DEF. XIII. The *Center of Gravity* of any body or system of bodies, is a point upon which the body or system, acted upon only by the force of gravity, will balance itself in all positions.

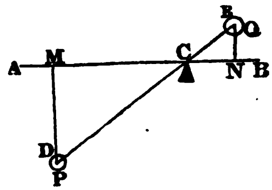
It will be made to appear that in every system there is such a point, by shewing how it may be found in every case. And it will also appear that there is only one point to which the definition is applicable.

Many of the properties of the point which we call the center of gravity, do not depend upon the action of gravity, and might be enunciated and proved without supposing that force to exist. This point has been by some authors called the *center of magnitude*, and by others, the *center of parallel forces.*

The definition given above supposes the particles of the system to be connected inflexibly; but the point may be found

by the same rule, when the particles are detached from one another.

82. Prop. *To find the center of gravity of two heavy particles.*



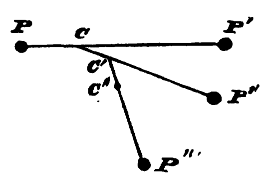
Let P, Q be two heavy particles. Join PQ, and divide PQ in C, so that weight of P : weight of Q :: CQ : CP. C will be the center of gravity.

For draw ACB horizontal and PM, QN vertical, meeting AB. If P, Q represent the weight of P and Q, we have $P : Q :: CQ : CP$; But $CQ : CP :: CN : CM$, by similar triangles. Hence, $P : Q :: CN : CM$, and $P \cdot CM = Q \cdot CN$. Wherefore P, Q balance (Art. 42): and this is true in all positions of P, Q: therefore C is the center of gravity.

Cor. 1. If two particles P, Q balance upon a point C in PQ in one position, they balance upon the same point in all positions.

Cor. 2. Also, in every position, if P, Q balance upon the point C, the pressure upon the point C is equal to the sum of the weights $P + Q$.

83. Prop. *To find the center of gravity of any number of heavy particles.*



Let P, P', P'', P''', be any heavy particles. Join PP'; and divide it in C, so that $P : P' :: CP' : CP$. Join CP''; and divide it in C', so that $P + P' : P'' :: C'P'' : C'C$. Join $C'P'''$; and divide it in C'', so that $P + P' + P'' : P''' :: C''P''' : C''C'$. And so on. The point C'' thus last found will be the center of gravity of $PP'P''P'''$.

For, P, P', balance upon C in every position, and produce upon C a pressure $P + P'$, by Cor. 1 and 2 of last Article. Hence the three particles P, P', P'' in every position will balance upon a point in CP'' in the same manner as if there were a particle $P + P'$ at C, and P'' at P''. Therefore, by last Article, they will in every position balance upon C', found as

in the construction. And the pressure upon C' in every position will be $P + P' + P''$. Hence the four particles P, P', P'', P''', in every position will balance upon a point in $C'P'''$, in the same manner as if there were a particle $P + P' + P''$ at C', and P''' at P'''. Therefore these will in every position balance upon C''. And so on.

And thus all the points P, P', P'', P''', will in every position balance upon C''; and C'' is the center of gravity.

Cor. 1. In every position of $PP'P''P'''$ the pressure upon the center of gravity C'' is equal to the sum of the weights of P, P', P'', P'''.

Cor. 2. Every system of heavy particles has a center of gravity: for the above construction is always possible.

84. Prop. *If a straight line pass through the center of gravity of a body or system, the body or system will balance itself on this line in all positions.*

For the body balances itself about the center of gravity, and is supported if that point be supported: but the point will be supported if a line passing through it be supported; and hence, if such a line be supported, the body will balance about it in all positions.

85. Prop. *If a system of heavy particles balance upon a straight line in all positions, the center of gravity is in that line.*

Let HK be a line upon which the system balances in all positions, and if possible, let G, the center of gravity, not be in HK. Draw FG parallel to HK. Then it will be possible to put the system in such a position that FG and HK are not in the same vertical plane. Draw vertical planes through FG and HK, and let x be the distance of these planes: and P, P', &c. being the particles of the body, let p, p', &c. be the perpendiculars drawn from the particles upon the vertical plane passing through FG: wherefore, P, P', &c. being on the same side as G, with respect to the vertical plane passing through HK, $p + x, p' + x$, &c. are the perpendiculars on this vertical plane. And if perpendiculars on the other side of the vertical plane be considered negative, these expressions are true for all the particles.

Since the system balances about the line *HK*, we have

$$P(p+x)+P'(p'+x)+\&c.=0.$$

But the system balances on the line *FG*, by last Article. Hence,

$$Pp+P'p'+\&c.=0.$$

Subtracting this equation from the former, we have

$$Px+P'x+\&c.=0,$$

which is impossible except $x=0$. That is, *FG*, *HK* are not in two different vertical planes; which is against the supposition. Therefore *FG* is not separate from *HK*. Q.E.D.

Cor. For vertical planes in which the center of gravity is, $Pp+P'p'+\&c.=0$. This is the *property of the center of gravity.*

86. Prop. *Any body or system will have the same effect in producing equilibrium by its weight, as if it were collected at its center of gravity.*

Let P, P', P'' be the particles of any system which by its weight produces equilibrium about any axis. Let a vertical plane be drawn through the axis about which the system is moveable; and another vertical plane, parallel to this, through the center of gravity; and let x be the distance of these vertical planes. Also let p, p', p'', &c. be the perpendiculars drawn from the particles P, P', P'' upon the vertical plane passing through the center of gravity; those perpendiculars being positive which are on the side farthest from the axis of motion. And let the moment of the force which is balanced by the system P, P', P'' about the axis be Qq. Therefore

$$Qq=P(p+x)+P'(p'+x)+P''(p''+x)+\&c.$$
$$=Pp+P'p'+P''p''+\&c.\ (P+P'+P''+\&c.)x.$$

But $Pp+P'p'+P''p''+\&c.=0$, by the property of the center of gravity (Art. 42);

Hence $Qq=(P+P'+P''+\&c.)x$;

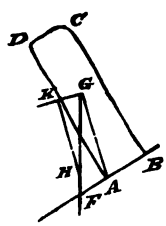
which is the same equation of equilibrium that we should have, if the particles P, P', P'', &c. were all situated in the center of gravity.

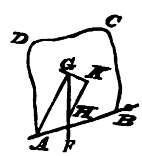
COR. The effect of a body to disturb equilibrium is the same as if it were collected in its center of gravity.

87. DEF. XIV. The *Base* of a heavy body is a side of it, touching another body, on which its direct pressure is supported.

The base is *bounded* by the lines, round which, or, if they are curves, round the tangents to which, the body must turn if it falls

88. PROP. *When a body touches with its base any surface, on which sliding is prevented, it will stand or fall according as the vertical drawn from its center of gravity falls within or without the base.*





Let *ABCD* be the body, *AB* its base, *G* its center of gravity. Join *AG*, then, in order that the body may fall over on the side *A*, *AG* must turn round the point *A* from the side on which the base *AB* is: therefore the point *G* must move in the direction *GK* perpendicular to *AG*, towards the side opposite to *AB*. If the vertical line *GF* fall without *AB*, take, in *GF*, *GH* to represent the weight of the body (which by Art. 86 produces the same effect as if it were collected in *G*), and draw *HK* perpendicular on *GK*. The force *GH* may be resolved into *GK*, *KH*, of which *KH* produces pressure at *A*, and is hence supported; and *GK* is unsupported, and produces motion in the direction *GK*; that is, causes the body to fall.

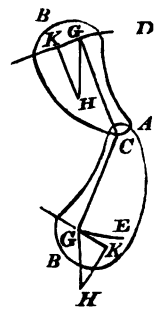
But if *GF* fall on the side of *A* on which *AB* is, *GH*, *HK* being drawn as before, the force *GH* may be resolved into *GK*, *KH*; of which *KH* is, as before, supported at *A*; and *GK* tends to turn the body in the direction *GB*, and is counteracted by the rigidity of the body. Therefore in this case the body will not fall.

Therefore the body will fall or stand as *GF* falls without or within *AB*.

89. PROP. *If a rigid heavy body be moveable about a fixt point, it will rest only when its center of gravity is at the highest or at the lowest point of its path.*

Let *AB* be a body moveable about *C*. If a fixt ver-

tical plane pass through C and the center of gravity, that center will describe a circle in this vertical plane; and if a vertical diameter be drawn, this diameter will determine the highest and lowest points of the path, D and E.



The heavy body produces the same effect as if it were collected at its center of gravity. Hence, if the center of gravity be at D or at E, there will be equilibrium; for in that case the whole weight will act in the line DC, or CE, and will be supported at C.

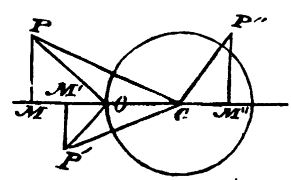
But if the center of gravity be in any other position, as G, draw GH vertical to represent the weight, GK perpendicular to CG, and HK parallel to CG. The force GH is equivalent to GK, KH; of which KH is supported by the fulcrum at C, and GK is not counteracted, and tends to make G move in the direction GK, in which direction it is moveable. Hence the body will move, and will not rest.

Cor. 1. It appears by the diagram that when the center of gravity is near the highest point, it tends to move from that position; and when it is near the lowest point, it tends to move towards that position.

Cor. 2. If the part of the body near to C be connected with the center of motion C, not by a rigid, but by a flexible portion, (as when a body hangs by a string from a fixt point) the equilibrium, when the center of gravity is at the highest point, is no longer possible. For the flexible portion cannot support the pressure in the direction DC In this case the body cannot rest except the center of gravity be at the lowest point.

90. Prop. *In any system of particles, the sum of the products formed by multiplying each particle into the square of its distance from a certain point, is least when the point is the center of gravity.*

Let O be the point, P, P', P'' the particles, G the center of gravity. Join OG, and draw PM, $P'M'$, $P''M''$, perpendicular upon OG. Then we have, by the properties of triangles,



$$PO^2 = PG^2 + GO^2 - 2GO.GM,$$
$$P'O^2 = P'G^2 + GO^2 - 2GO.GM',$$
$$P''O^2 = P''G^2 + GO^2 - 2GO.GM'',$$

the quantities GM'', &c. being reckoned negative when M'' is on the other side of G. Hence,

$$P.PO^2 + P'.P'O^2 + P''.P''O^2 =$$
$$P.PG^2 + P'.P'G^2 + P''.P''G^2 + (P + P' + P'')GO^2$$
$$- 2GO(P.GM + P'.GM' + P''.GM'').$$

But $P.GM + P'.GM' + P''.GM'' = 0$, on the above supposition; therefore

$$P.PO^2 + P'.P'O^2 + P''.P''O^2$$
$$= P.PG^2 + P'.P'G^2 + P''.P''G^2 + (P + P' + P'')GO^2;$$

hence the sum on the first side of the equation is least when GO vanishes, that is, when O coincides with G.

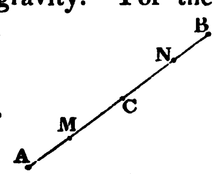
Cor. If with center G and any radius GO we describe a circle, the sum $P.PO^2 + P'.P'O^2 + P''.P''O^2$ is the same for every point in this circle. For GO is the same.

91. Prop. *To find the center of gravity of given figures.*

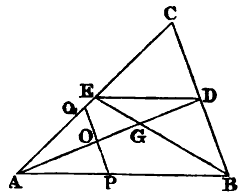
The modes of solving this problem will appear in the following examples.

Ex. 1. The center of gravity of a uniform material line AB.

Bisect AB in C; C is the center of gravity. For the line may be supposed to be made up of equal material particles: and if CM, CN be taken equal, the particles M, N will balance upon C in every position; and the same is true of any other pair of particles; therefore the whole line, which is made up of these particles, will balance upon C in every position, and C is the center of gravity.



Ex. 2. The center of gravity of a uniform material plane triangle.



Let ABC be the triangle: bisect BC in D; join AD; take $DG = \frac{1}{3} AD$; G is the center of gravity of the triangle.

For the whole triangle may be conceived to be made up of material lines parallel to BC, as PQ; and each of these will be bisected by the line AD, as in O; and therefore will balance upon the point O in all positions. Therefore the triangle ABC will balance upon the line AD in all positions: and therefore (Art. 85) the center of gravity is in the line AD. For the same reason the center of gravity is in the line BE, E being the middle point of AC. Therefore the center of gravity is in the point G, the intersection of these two lines.

Join DE, then, by similar triangles,

$$DE : AB :: CD : CB :: 1 : 2.$$

$$\text{Hence } DG : GA :: DE : AB :: 1 : 2;$$

$$\text{whence } AG = 2\,DG,\ AD = 3\,DG,\ DG = \tfrac{1}{3} AD.$$

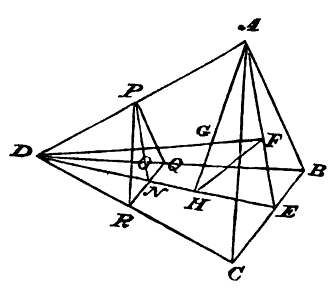
Ex. 3. To find the center of gravity of any polygon.

The polygon may be divided into triangles; the center of gravity of each of these triangles may be found by Ex. 2; and the weight of each triangle being supposed to be collected at its center of gravity, the center of gravity of the system of particles may be found by Art. 83.

It is manifest that in similar polygons, the centers of gravity will be similarly situated.

Ex. 4. The center of gravity of a uniform triangular pyramid.

Let ABC be the triangular base, and D the vertex of the pyramid. Bisect BC in E; join AE, and take $EF = \frac{1}{3} AE$; join DF and take $FG = \frac{1}{4} DF$; G is the center of gravity of the pyramid.



For the whole pyramid may be conceived to be made up of material planes parallel to ABC, as PQR; and if the line DF meets this in O, and if PO meets

QR in N; it may easily be shewn that because $BE = EC$, QN is $= NR$; and because $EF = \frac{1}{3} AE$, NO is $= \frac{1}{3} NP$. Therefore O is the center of gravity of the triangle PQR, and this triangle will balance about the point O in all positions. Therefore the pyramid, made up of such triangles, will balance about the line DF in all positions: therefore the center of gravity of the pyramid is in the line DF.

For the same reason, if we take $EH = \frac{1}{3} ED$ and join AH, the center of gravity of the pyramid is in the line AH. Hence, the center of gravity of the pyramid is in the point G, the intersection of DF, AH.

Join FH. By similar triangles,

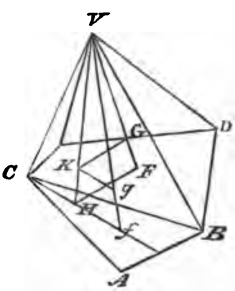
$$FH : AD :: EF : AE :: 1 : 3.$$

Hence $FG : GD :: FH : AD :: 1 : 3$;

whence $GD = 3FG$, $DF = 4FG$, $FG = \frac{1}{4} DF$.

Ex. 5. The center of gravity of any pyramid.

Let V be the vertex, and $ABCD$ the base, of any pyramid. Take F the center of gravity of the base $ABCD$; join VF, and take $FG = \frac{1}{4} VG$: G will be the center of gravity of the pyramid.



For the base may be divided into triangles; let ABC be one of these triangles, f its center of gravity, Vgf meeting the base, gf being $\frac{1}{4} Vf$. Let VH be drawn perpendicular to the base, gK parallel to fH. Then it is easily seen that $KH = \frac{1}{4} VH$. And the plane parallel to the base, passing through g, meets VH in K.

Hence the plane gKG, which, being parallel to the base, passes through the center of gravity of a triangular pyramid, cuts the line VH, drawn perpendicular to the base, so that $KH = \frac{1}{4} VH$.

And this is the case for each of the pyramids having triangular bases, of which the whole pyramid is composed: all their centers of gravity lie in the plane parallel to the base and passing through K. Therefore the center of gravity of the whole is in this plane.

But it is also in the line VF: for the pyramid may be supposed to be made up of material planes, parallel to the base, which will all be similar to the base; and the line VF will meet each of these in its center of gravity, because the centers of gravity of similar polygons are similarly situated. Hence the pyramid will balance upon the line VF in all positions, and its center of gravity is in that line.

Hence the center of gravity of the pyramid is in the point G where the plane gKG meets VF; and hence $FG = \frac{1}{4} VF$.

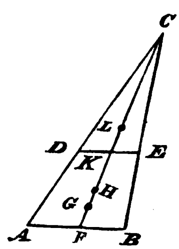
Ex. 6. The center of gravity of a cone.

A cone may be considered as a pyramid having for its base a polygon which, at the limit, becomes a curve. Hence the same construction is applicable in this case as in the case of a pyramid.

If an axis be drawn from the vertex to the center of gravity of the base, the center of gravity of the cone is in this axis; at a distane of $\frac{1}{4}$ of the axis from the base, and therefore $\frac{3}{4}$ of the axis from the vertex.

Ex. 7. The center of gravity of a trapezium with two parallel sides.

Let $ABDE$ be a trapezium of which the sides AB, DE are parallel. Produce AD, BE to meet in C, bisect AB in F; join CF, meeting DE in K. It is easily shewn that since $AF = FB$, DK is $= KE$. Take $FH = \frac{1}{3} CF$, and $KL = \frac{1}{3} CK$; then H will be the center of gravity of the triangle ABC, and L will be the center of gravity of the triangle CDE. Let G be the center of gravity of the trapezium $ABDE$.



The triangle ABC is made up of the trapezium $ABDE$ and the triangle CDE. Hence, if each of these figures be collected in its center of gravity at G and L, and the center of gravity of the two particles G, L be found, this must be H, the center of gravity of ABC. Hence the moment round C, of the whole triangle ABC collected in H, must be equal to the moments of G and L: that is,

$$\text{triangle } CAB \,.\, CH = \text{trapezium } ABDE \,.\, CG$$
$$+ \text{ triangle } CDE \,.\, CL.$$

Let $AB = a$, $DE = b$, $KF = c$, $CF = x$, $FG = y$;

triangle $ABC = A$.

Then triangle CDE = triangle $ABC \cdot \dfrac{DE^2}{AB^2} = A \cdot \dfrac{b^2}{a^2}$,

trapezium $ABDE$ = triangle ABC − triangle CDE

$$= A\left(1 - \frac{b^2}{a^2}\right).$$

$$CH = \frac{2}{3}x,\ CL = \frac{2}{3}(x - c),\ CG = x - y,$$

Hence $A \cdot \dfrac{2}{3}x = A\left(1 - \dfrac{b^2}{a^2}\right)(x - y) + A\dfrac{b^2}{a^2}\dfrac{2}{3}(x - c)$;

whence $3(a^2 - b^2)\,y = (a^2 - b^2)\,x - 2b^2c$.

But by the figure, $x : x - c :: a : b$; whence $x = \dfrac{ac}{a - b}$;

$3(a^2 - b^2)\,y = a^2c + abc - 2b^2c$; and dividing by $a - b$,

$$3(a + b)\,y = ac + 2bc;$$

Hence $y = \dfrac{c}{3} \cdot \dfrac{a + 2b}{a + b}$.

Ex. 8. The center of gravity of a frustum of a pyramid.

In this case we may proceed nearly as in the last, F, K being the centers of gravity of the two ends of the frustum, $FH = \frac{1}{4}CF$, $KL = \frac{1}{4}CK$. Also if A be the whole pyramid, a, b any homologous lines in the two ends, c the line FK, $CF = x$, $FG = y$; we have, pyramid $CDE = A\dfrac{b^3}{a^3}$; frustum $BD = A\left(1 - \dfrac{b^3}{a^3}\right)$; pyramid $CAB \cdot CH$ = frustum $BD \cdot CG$ + pyramid $CDE \cdot CL$. Whence

$$A \cdot \frac{3}{4}x = A\left(1 - \frac{b^3}{a^3}\right)(x - y) + A\frac{b^3}{a^3}\frac{3}{4}(x - c).$$

Hence $4(a^3 - b^3)\,y = (a^3 - b^3)\,x - 3b^3c$.

But, as before, $x = \dfrac{ac}{a-b}$; hence

$$4(a^3 - b^3)\,y = (a^2 + ab + b^2)\,ac - 3b^3c$$
$$= (a^3 + a^2b + ab^2 - 3b^3)\,c.$$

Dividing both sides by $a - b$.

$$4(a^2 + ab + b^2)\,y = (a^2 + 2ab + 3b^2)\,c.$$

$$y = \frac{c}{4}\cdot\frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

The same formula will hold for a frustum of a cone, a, b being the radii of the greater and smaller ends.

92. Prop. *To find the center of gravity of any body defined by an equation among rectangular co-ordinates.*

Let one of the co-ordinates of any point in the body be measured along an axis x; and let m be the mass of the body cut off by a plane perpendicular to the abscissa at the distance x; m' the mass cut off by the plane perpendicular to x at the distance of a succeeding value x'. Also let h be the abscissa of the center of gravity for the mass m, h' the abscissa of the center of gravity for m'; and k the abscissa of the center of gravity of the mass $m' - m$, included between the two planes perpendicular to x. Then, since the whole mass m' may be conceived to be made up of the parts m and $m' - m$, we have, taking the moments about the origin of x,

$$m'h' = mh + (m' - m)\,k, \text{ whence } k = \frac{m'h' - mh}{m' - m}.$$

Now when the difference of x' and x vanishes, and we pass to the limit, this becomes $k = \dfrac{d\,.\,mh}{d\,.\,m}$.

But in this case the thickness of the portion in the direction of x vanishes; x' and x become ultimately equal; the quantity k, which is always between them, becomes also x, and the equation becomes

$$x = \frac{d\,.\,mh}{d\,.\,m} = \frac{\dfrac{d\,.\,mh}{dx}}{\dfrac{d\,.\,m}{dx}}.$$

Whence $x\dfrac{dm}{dx} = \dfrac{d\,.\,mh}{dx}$.

Hence, integrating with respect to x,

$$\int_x x\frac{dm}{dx} = mh\,;\ h = \frac{\int_x x\dfrac{dm}{dx}}{m}.$$

Or, since m may also be found by integration, and is $= \int_x \dfrac{dm}{dx}$,

$$h = \frac{\int_x x\dfrac{dm}{dx}}{\int_x \dfrac{dm}{dx}}.$$

In like manner we may find the other co-ordinate, or the two other co-ordinates of the center of gravity, according as the figure is of two or of three dimensions. The three co-ordinates being h, k, l, we have

$$h = \frac{\int_x x\dfrac{dm}{dx}}{\int_x \dfrac{dm}{dx}},\quad k = \frac{\int_y y\dfrac{dm}{dy}}{\int_y \dfrac{dm}{dy}},\quad l = \frac{\int_z z\dfrac{dm}{dz}}{\int_z \dfrac{dm}{dz}}.$$

The value of $\dfrac{dm}{dx}$ in terms of the co-ordinates and their differentials is known by the Differential Calculus; and this being substituted, the integration may be performed.

If the figure be a plane curve, $\dfrac{dm}{dx} = y$.

If it be the arc of a plane curve, $\dfrac{dm}{dx} = \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}$.

If it be a solid of revolution, $\dfrac{dm}{dx} = \pi y^2$.

If it be a surface of revolution, $\dfrac{dm}{dx} = 2\pi\sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}$.

If it be any solid, $\dfrac{dm}{dx} = \int_y z$. Also $\dfrac{d^2m}{dx\,dy} = z$.

And we may put the formula in this shape, $h = \dfrac{\int_x \int_y x z}{\int_x \int_y}$.

If the figure be any curve surface, and if $\dfrac{dz}{dx} = p$, $\dfrac{dz}{dy} = q$, we have $\dfrac{d^2 m}{dx\, dy} = \sqrt{(1 + p^2 + q^2)}$; whence h may be found, as in the last case.

Examples of finding the center of gravity.

Ex. 1. Let the body be a parabolic conoid; and let x be measured from the vertex. Then, $4a$ being the parameter,

$$\frac{dm}{dx} = \pi y^2 = 4\pi a x, \quad \int_x x \frac{dm}{dx} = 4\pi a \int_x x^2 = \tfrac{4}{3}\pi a x^3,$$

which begins when $x = 0$, as it should do.

Also for the denominator,

$$\int_x \frac{dm}{dx} = \int_x 4\pi a x = 2\pi a x^2.$$

Hence $h = \frac{2}{3}x$.

Ex. 2. Let the body be a segment of a sphere; and let x be measured from the vertex of the segment. Then, a being the radius,

$$\frac{dm}{dx} = \pi\,(2ax - x^2), \quad \int_x x \frac{dm}{dx} = \pi \int (2ax^2 - x^3) = \pi\left(\frac{2}{3}ax^3 - \frac{1}{4}x^4\right),$$

which begins when $x = 0$, as it should do.

Also the denominator,

$$\int_x \frac{dm}{dx} = \int_x \pi(2ax - x^2) = \pi\left(ax^2 - \frac{1}{3}x^3\right).$$

$$\text{Hence } h = \frac{8a - 3x}{3a - x} \cdot \frac{x}{4}.$$

For the whole hemisphere, when $x = a$, $h = \dfrac{5a}{8}$.

CHAPTER VI.

PROBLEMS CONCERNING EQUILIBRIUM.

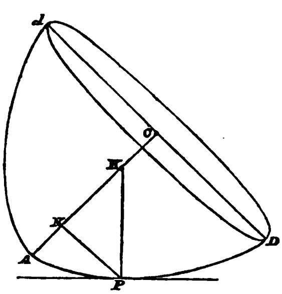
The following Problems may serve to exemplify the principles contained in the last two Chapters.

1. *Equilibrium on a Surface.*

93. When a body rests on a given surface, it will touch it either in one point, or in several points, or with a finite portion of its surface. In all these cases the body must be supposed to be acted on by forces perpendicular to the surface at the points where it is in contact; that is, by the re-action of the surface at those points.

Prob. I. *A Parabolic Conoid* DAd *rests upon a horizontal plane; to find its position.*

If PK be a vertical line drawn through the point of contact, meeting the axis in K, this line must pass through the center of gravity; for the body may be supposed to be collected in its center of gravity, and it will then be supported by the re-action which acts in the line PK. And since the center of gravity is a point in the axis, K must be this center. Also, since PK is perpendicular to the tangent at P,



PK is a normal; and hence if PN be perpendicular to the axis, by Conic Sections, $NK = \frac{1}{2}$ parameter $= 2a$.
And

$$\tan . AKP = \frac{NP}{NK} = \frac{\sqrt{4a \, . \, AN}}{2a} = \sqrt{\frac{AN}{a}} = \sqrt{\frac{AK - 2a}{a}}$$

$$= \sqrt{\left\{\frac{AK}{a} - 2\right\}}.$$

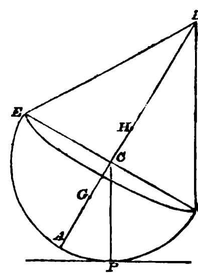
If AK be less than $2a$, this answer is impossible, that is, there will no longer be an oblique position of equilibrium, and the figure will not rest except when the axis is vertical.

By Art. 92, Ex. 1, in a homogeneous parabolic conoid, if K be the center of gravity, $AK = \frac{2}{3}AC$. Hence the oblique position of equilibrium is possible so long as $\frac{2}{3}AC > 2a$, or $AC > 3a$.

Prob. II. *A solid composed of a Cone and a Hemisphere on the same base rests on a horizontal plane; to find its dimensions that it may rest on the hemispherical end in all positions.*

PC, the vertical line, will, in all positions, meet the axis in the center of the sphere; and hence this point must be the center of gravity of the whole figure. Let G be the center of gravity of the hemisphere, and H of the cone; and we must have,

$$\text{mass of cone} \times CH$$
$$= \text{mass of hemisphere} \times CG.$$



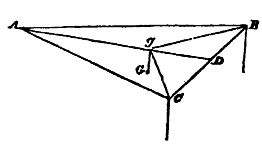
The cone is $\frac{1}{3}$, and the hemisphere is $\frac{2}{3}$, of the circumscribing cylinder. Hence cone $=$ base $DE \times \frac{1}{3} BC$, and the hemisphere $=$ base $DE \times \frac{2}{3} AC$. Also by Art. 91, Ex. 6, $CH = \frac{1}{4}BC$; and by Art. 92, Ex. 2, $AC = \frac{5}{8}CA$, and $CG = \frac{3}{8}AC$. Hence base $DE \times \frac{1}{3} BC \times \frac{1}{4} BC =$ base $DE \times \frac{2}{3} AC \times \frac{3}{8} AC$;

$$\therefore\ BC^2 = 3AC^2.$$

Hence $BD^2 = BC^2 + CD^2 = 4AC^2$; and $BD = 2AC = DE$. Hence the triangle DBE is equilateral.

Prob. III. *When a body is supported on three vertical props* (A, B, C); *to find the pressure on each.*

Let G be the center of gravity of the body, Gg a vertical line meeting the plane ABC in g; join Ag, meeting BC in D; then, if we suppose the whole mass collected at the center of gravity, it may be considered as supported on a lever AD; and if W be the whole weight;



pressure at A : W :: Dg : DA :: triangle BgC : triangle BAC.

In the same manner,

pressure at B (or C) : W :: triangle AgC (or AgB) : triangle BAC.

Hence the pressure on each prop is as the triangle opposite to it, made by joining the angles of the triangle ABC with the point g.

Cor. When a body is supported on four vertical props, as a table on its four legs, the pressures will be indeterminate, if we consider the body as perfectly rigid. For since it may be supported on three of these props, the fourth may support either nothing, or a finite portion of the weight. The only conditions are, that the pressures have their sum equal to the weight of the body, and that they be such, that if they be considered as weights, their center of gravity and the center of gravity of the body are in the same vertical line.

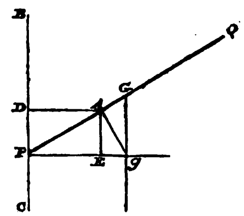
The same is true if the props be more than four.

2. *Equilibrium on a Point and a Surface.*

94. When a body rests with one part of it upon a point, and another upon a surface, the forces by which it is supported are the re-actions of the point and of the surface. If A be the supporting point, the re-action will be in Ag perpendicular to the surface of the body. (See next page.) And if PB be the supporting surface, and P the point of contact, the re-action there will be in Pg perpendicular to both the surfaces. And by Art. 66, the point g of intersection of the two forces must be in the line in which the third force acts; that is, in the vertical line passing through G the center of gravity. Hence Gg is vertical; and from this property the position of equilibrium may be determined.

Prob. IV. *A given beam* PQ, *considered as a straight line, rests upon a given point* A, *with its end against a vertical plane* BC; *to find the position in which it will rest.*

Let G be the center of gravity, Pg horizontal, Ag perpen-



dicular to PA, Gg joined; and since the re-action of the plane is in Pg, and that of the point A in gA, the point of intersection g, of these forces, must be in the vertical line through the center of gravity: hence Gg is vertical, and perpendicular to Pg. Draw AE vertical. By the Elements of Geometry, the triangles PAE, PgA, PGg, are all similar; hence

$$\frac{PE}{PA} = \frac{PA}{Pg},$$

$$\frac{PE}{PA} = \frac{Pg}{PG}; \quad \therefore \frac{PE^2}{PA^2} = \frac{PA}{PG};$$

$\therefore PA^3 = PG \,.\, PE^2 = PG \,.\, AD^2$; $\therefore PA = \sqrt[3]{(PG \,.\, AD^2)}$, and is therefore known; for PG and AD are known.

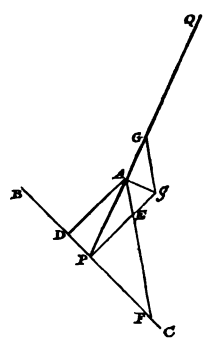
Cor. 1. To find the inclination of PQ, we have

$$\cos PAD = \frac{AD}{PA} = \sqrt[3]{\frac{AD^3}{AG \,.\, AD^2}} = \sqrt[3]{\frac{AD}{PG}}.$$

Cor. 2. Hence if AD be greater than PG the equilibrium is impossible.

Prob. V. *The same suppositions remaining, except that the plane* BC *is now inclined to the horizon at any angle, to find the position of the equilibrium.*

Let, as before, G be the center of gravity, Pg perpendicular to BC, and Ag to PA, and therefore Gg vertical, and also let AEF be vertical, and AD perpendicular to BC. We have, by similar triangles,



$$\frac{FP}{FD} = \frac{PE}{AD},$$

$$\frac{PG}{Pg} = \frac{PA}{PE},$$

$$\frac{Pg}{PA} = \frac{PA}{AD}; \text{ and multiplying,}$$

$$\frac{FP \cdot PG}{FD \cdot PA} = \frac{PA^2}{AD^2}; \quad \therefore \frac{PA^3}{PF} = \frac{PG \cdot AD^2}{FD}.$$

Let $PG = a$, $AD = b$, $FD = c$; $PA = x$;

$$\therefore PD = (x^2 - b^2)^{\frac{1}{2}}, \quad PF = c - (x^2 - b^2)^{\frac{1}{2}};$$

hence we have

$$\frac{x^3}{c - (x^2 - b^2)^{\frac{1}{2}}} = \frac{ab^2}{c};$$

whence x must be found.

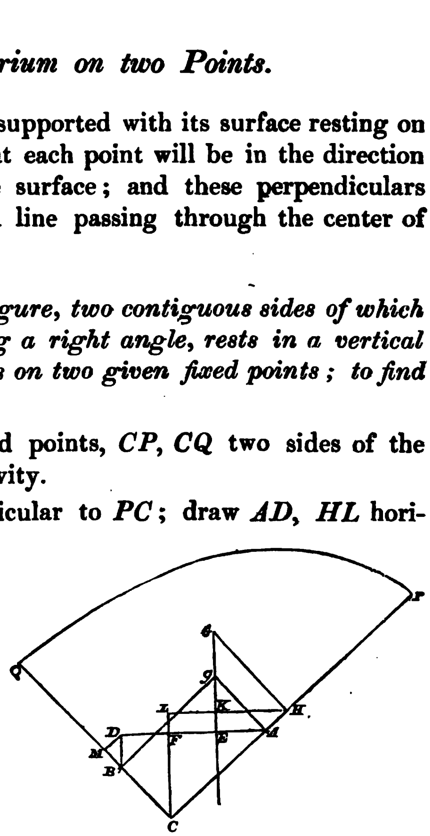
3. *Equilibrium on two Points.*

95. When a body is supported with its surface resting on two points, the re-action at each point will be in the direction of a perpendicular to the surface; and these perpendiculars must meet in the vertical line passing through the center of gravity as before.

PROB. VI. *A plane figure, two contiguous sides of which are straight lines forming a right angle, rests in a vertical plane with these two sides on two given fixed points; to find its position.*

Let A, B be the fixed points, CP, CQ two sides of the figure, G its center of gravity.

Let GH be a perpendicular to PC; draw AD, HL horizontal, BD, CFL, GKE vertical. And if Ag and Bg be perpendicular to the sides CA, CB, they will be in the directions of the pressures exerted at A and B on their sides; and these directions will meet in the vertical line passing through G. Draw also DM perpendicular to BC.



Let $AD = a$, $BD = b$, $CH = h$, $HG = k$; and let the angle CAD be θ: then DBM, CHL, HGK also $= \theta$.

And since $AgBC$ is a rectangle, $Ag = CB$; and hence it appears that

$$AE = DF; \quad \therefore \; EF = AD - 2FD.$$

Also $BC = CM - BM = a \sin\theta - b\cos\theta$;

$$\therefore \; FD = BC \sin\theta = a\sin^2\theta - b\cos\theta\sin\theta;$$

$$\therefore \; EF = AD - 2FD = a - 2a\sin^2\theta + 2b\sin\theta\cos\theta.$$

But $EF = KL = HL - HK = h\cos\theta - k\sin\theta$;

$$\therefore \; a - 2a\sin^2\theta + 2b\sin\theta\cos\theta = h\cos\theta - k\sin\theta;$$

$$\text{or} \quad a\cos 2\theta + b\sin 2\theta = h\cos\theta - k\sin\theta;$$

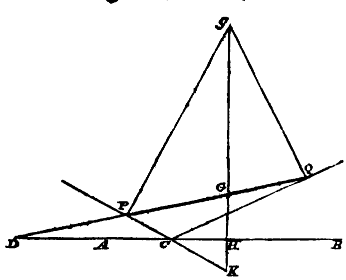
from which equation θ is to be determined.

4. *Equilibrium on two Surfaces.*

96. When a body rests on two surfaces, the re-actions at the points of support will take place in lines perpendicular to these surfaces; these lines must meet, for otherwise the body cannot be supported. And as before, the point of concourse will be in the vertical passing through the center of gravity.

PROB. VII. *A given beam* PQ, *considered as a line, is supported on two given inclined planes* CP, CQ; *to find the position in which it will rest.*

Let Pg, Qg, perpendicular to the planes, meet in g; and G being the center of gravity of PQ, Gg will be vertical. Let gG meet the horizontal line drawn through C, in H, and the plane PC in K. The angle PgK is the complement of PKg, as is also KCH. Therefore PgG is equal to KCH or PCA; similarly, QgG is equal to QCB.



Let PCA, the inclination of the plane PA, $= \iota$, $QCB = \iota'$; $\therefore PgG = \iota$, $QgG = \iota'$; also let $PG = a$, $QG = a'$, and let QP produced meet the horizontal plane in D, and let $PDC = \delta$:

$$\text{hence } CPQ = PCD + CDP = \iota + \delta,$$

$$CQP = QCB - QDC = \iota' - \delta.$$

Now

$$\frac{PG}{Gg} = \frac{\sin PgG}{\sin GPg},$$

$$\frac{Gg}{QG} = \frac{\sin GQg}{\sin QgG};$$

$$\therefore \frac{PG}{QG} = \frac{\sin PgG}{\sin QgG} \cdot \frac{\sin GQg}{\sin GPg} = \frac{\sin PgG}{\sin QgG} \cdot \frac{\cos PQC}{\cos QPC},$$

$$\text{or } \frac{a}{a'} = \frac{\sin \iota}{\sin \iota'} \cdot \frac{\cos (\iota' - \delta)}{\cos (\iota + \delta)}$$

$$= \frac{\sin \iota}{\sin \iota'} \cdot \frac{\cos \iota' . \cos \delta + \sin \iota' . \sin \delta}{\cos \iota . \cos \delta - \sin \iota . \sin \delta}$$

$$= \frac{\tan \iota}{\tan \iota'} \cdot \frac{1 + \tan \iota' . \tan \delta}{1 - \tan \iota . \tan \delta}.$$

Whence

$$a \tan \iota' - a \tan \iota . \tan \iota' . \tan \delta = a' . \tan \iota + a' . \tan \iota . \tan \iota' . \tan \delta;$$

$$\therefore \tan \delta = \frac{a \tan \iota' - a' \tan \iota}{(a + a') \tan \iota \tan \iota'} = \frac{a \operatorname{cotan} \iota - a' \operatorname{cotan} \iota'}{a + a'};$$

whence we know the inclination of PQ to the horizon.

COR. 1. If $a = a'$, which it will be, if the line PQ be of uniform thickness and density;

$$\tan \delta = \frac{\tan \iota' - \tan \iota}{2 \tan \iota \tan \iota'} = \frac{1}{2} \cdot \left(\frac{1}{\tan \iota} - \frac{1}{\tan \iota'}\right);$$

$$= \frac{\operatorname{cotan} \iota - \operatorname{cotan} \iota'}{2}.$$

COR. 2. If $\iota' = \iota$, or the planes be equally inclined,

$$\tan \delta = \frac{a - a'}{a + a'} . \operatorname{cotan} \iota.$$

COR. 3. In order that PQ may rest parallel to the horizon, we must have $\delta = 0$;

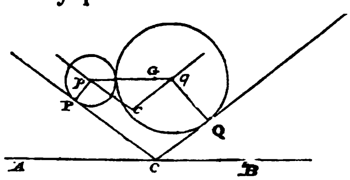
$$\therefore\ a \tan \iota' - a' \tan \iota = 0;$$

$$\therefore\ \frac{a}{a'} = \frac{\tan \iota}{\tan \iota'};$$

the segments GP, GQ must be as the tangents of the inclinations.

97. PROB. VIII. *Let* p, q *be two Spheres, touching each other and resting on two inclined planes* CP, CQ; *to find their position.*

Join p, q, their centers. In every position the distance of their centers is equal to the sum of their radii: and hence they have no tendency to change their point of contact with each other, and may be considered as one mass. Also the re-action is perpendicular to the planes which touch the spheres, and will therefore pass through the centers p, q. Hence pq will be supported in the same way as if it rested at p and q, on planes cp, cq, parallel to CP and CQ. Hence we may find its position by the last problem.



Let r and r' be the radii of the spheres, p and q their weights, and δ the inclination of pq to the horizon. Let G be the center of gravity of the mass pq, therefore we shall have, retaining the notation of the last problem,

$$pq = r + r',\ pG = \frac{(r + r')q}{q + p} = a,\ qG = \frac{(r + r')p}{p + q} = a';$$

$$\text{hence}\ \tan \delta = \frac{a \tan \iota' - a' \tan \iota}{(a + a') \tan \iota \tan \iota'},$$

$$\text{will} = \frac{q \tan \iota' - p \tan \iota}{(p + q) \tan \iota \tan \iota'}.$$

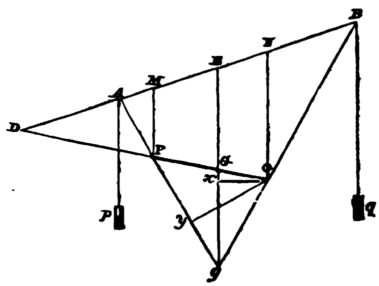
COR. Hence it appears that the inclination of pq is independent of the radii r, r', and depends only upon the weights of the spheres.

The effect will be exactly the same whether the body be supported by the re-action of a surface, or by the tension of a

string perpendicular to the surface. If any point of it hang by a string of given length, it will be confined to the surface of a sphere, and the case will be the same as if it rested on a spherical surface.

98. Prob. IX. *A given beam* PQ *hangs by two strings of given lengths* AP, BQ, *from two given fixed points* A, B; *to find its position when it rests.*

Let AP, BQ meet in g; therefore gG through the center of gravity G is vertical; let this meet AB in E, and let PM, QN be parallel to it; also let QP meet BA in D.



Let $AB = c$, and its inclination to the vertical, $AEG = \epsilon$; $AP = p$, $BQ = q$, $GP = a$, $GQ = b$; $PAB = \alpha$, $QBA = \beta$, $PDA = \delta$. Hence

$$gPQ = APD = PAB - PDA = \alpha - \delta,$$

$$gQP = QBD + QDB = \beta + \delta,$$

$$AgB = PgQ = \pi - (\alpha + \beta);$$

$$\therefore \; Ag = AB \cdot \frac{\sin ABg}{\sin AgB} = c \cdot \frac{\sin\beta}{\sin(\alpha+\beta)},$$

$$Bg = AB \cdot \frac{\sin BAg}{\sin BgA} = c \cdot \frac{\sin\alpha}{\sin(\alpha+\beta)};$$

$$\therefore \; Pg = Ag - AP = c \cdot \frac{\sin\beta}{\sin(\alpha+\beta)} - p;$$

$$Qg = Bg - BQ = c \cdot \frac{\sin\alpha}{\sin(\alpha+\beta)} - q:$$

but

$$Pg = PQ \cdot \frac{\sin PQg}{\sin PgQ} = (a+b)\frac{\sin(\beta+\delta)}{\sin(\alpha+\beta)};$$

$$Qg = QP \cdot \frac{\sin QPg}{\sin QPg} = (a+b)\frac{\sin(\alpha-\delta)}{\sin(\alpha+\beta)};$$

$$\text{hence } c \cdot \frac{\sin\beta}{\sin(\alpha+\beta)} - p = (a+b) \cdot \frac{\sin(\beta+\delta)}{\sin(\alpha+\beta)};$$

$$c \cdot \frac{\sin\alpha}{\sin(\alpha+\beta)} - q = (a+b) \cdot \frac{\sin(\alpha-\delta)}{\sin(\alpha+\beta)};$$

$$\text{or } c\sin\beta - p \cdot \sin(\alpha+\beta) = (a+b) \cdot \sin(\beta+\delta) \ldots\ldots (1),$$

$$c\sin\alpha - q \cdot \sin(\alpha+\beta) = (a+b) \cdot \sin(\alpha-\delta) \ldots\ldots (2).$$

To obtain α, β, δ, we must have a third equation; for this purpose we must find the tensions of the strings PA, QB; and as these tensions must be equivalent to the weight, which acts in a vertical direction, their components in a horizontal direction must destroy each other.

To find the tension of the string PA, we may suppose the point Q to be a fulcrum on which the beam PQ is sustained by the string PA; hence if we draw Qx and Qy perpendicular on Gg and Ag, we have

$$\frac{\text{tension of } PA}{\text{weight of } PQ} = \frac{Qx}{Qy}:$$

or, if we call the tensions of PA, QB, P, Q, and the weight of PQ, W; we shall have

$$\frac{P}{W} = \frac{Qx}{Qy} = \frac{QG \cdot \sin QGx}{QP \cdot \sin QPy}:$$

$$= \frac{QG \cdot \sin(GDE + GED)}{QP \cdot \sin(PAB - PDA)},$$

$$= \frac{b \cdot \sin(\delta+\epsilon)}{(a+b) \cdot \sin(\alpha-\delta)}.$$

Similarly, we should have

$$\frac{Q}{W} = \frac{a \cdot \sin(\delta+\epsilon)}{(a+b) \cdot \sin(\beta+\delta)}.$$

$$\text{Hence } \frac{P}{Q} = \frac{b \cdot \sin(\beta+\delta)}{a \cdot \sin(\alpha-\delta)}.$$

But the forces which draw the beam in the horizontal direction are the resolved parts of these tensions; that is, $P \sin APM$ and $Q \sin BQN$; $\therefore P \sin APM = Q \sin BQN$.

But $\sin APM = \sin(AMP + PAM) = \sin(\epsilon + a)$,

$$\sin BQN = \sin(ANQ - QBN) = \sin(\epsilon - \beta);$$

$$\therefore P = Q\,\frac{\sin(\epsilon - \beta)}{\sin(\epsilon + a)};$$

hence $$\frac{b \sin(\beta + \delta)}{a \sin(a - \delta)} = \frac{\sin(\epsilon - \beta)}{\sin(\epsilon + a)} \ldots\ldots\ldots (3).$$

And the three equations (1), (2), (3), will give the three unknown quantities a, β, δ.

COR. 1. If the center of gravity of PQ be in its middle point, which it will be if the beam be of uniform thickness and density, $a = b$; hence

$$\frac{P}{Q} = \frac{\sin(\beta + \delta)}{\sin(a - \delta)} = \frac{\sin BQP}{\sin APQ},$$

or the tensions are inversely as the signs of the angles at P and Q.

COR. 2. If A, B be in the same horizontal line, $\epsilon = \frac{\pi}{2}$, and equation (3) becomes

$$\frac{b \,.\, \sin(\beta + \delta)}{a \,.\, \sin(a - \delta)} = \frac{\cos\beta}{\cos a}.$$

99. PROB. X. *A beam* PQ *is supported by strings which go over given pullies* A, B, *and have given weights* P *and* Q *attached to them at* p *and* q; *to find its position.*

Let $PAB = a$, $QBA = \beta$, (fig. p. 73), and the rest of the notation as in the last problem: the tensions of the strings Ap, Bq must be equal to the weights P, Q: hence, by the expressions there found for the tensions;

$$\frac{P}{W} = \frac{b}{(a + b)} \cdot \frac{\sin(\delta + \epsilon)}{\sin(a - \delta)},$$

$$\frac{Q}{W} = \frac{a}{(a + b)} \cdot \frac{\sin(\delta + \epsilon)}{\sin(\beta + \delta)}.$$

Also, as before, the equation (3) of last problem must be satisfied;

$$\therefore \frac{b \sin (\beta + \delta)}{a \sin (\alpha - \delta)} = \frac{\sin (\epsilon - \beta)}{\sin (\epsilon + \alpha)}:$$

from which three equations α, β, δ, must be determined.

100. If a body be acted on by more than three forces in the same plane, we may suppose any two of them to be applied at their point of concourse. We may then suppose that at this point the resultant of the two forces is substituted for them: by this means the number of forces will be less by one than it was; and by successive operations of this kind we may reduce the forces to three, which is the case already considered.

APPENDIX TO THE STATICS.

CHAPTER VII.

ILLUSTRATIONS FROM THE FORMER EDITIONS.

Art. 3. *Forces may produce rest as well as motion.*

In fig. 1, if a person standing on the bank of a canal, as at *P*, pull a boat *B* which is in the water, by means of a rope *BP*, he will cause the boat to move in the direction *BP*; but if there be other persons, as at *Q*, *R*, also pulling the boat in the directions, *BQ*, *BR*, it may happen, by properly adjusting the directions of the ropes and the strength exerted, that the boat shall remain at rest by the united action of the three forces.

Art. 16. *Forces are measurable quantities.*

Instead of supposing weights to *draw out* a spring-balance, that is, a spiral spring, as is done in this Article, we may suppose the weights to *bend* an elastic rod, as *MA*, fig. 2. If *Q* bend the spring just as far as *P* does, namely, to the position *MB*, *Q* is equal to *P*. And if *P* and *Q* together bend the spring into the position *MC*, any weight which bends it into the position *MC* is $2P$. And if three such weights as *P* bend the spring into the position *MD*, any weight which bends it into the position *MD* is $3P$; and so on.

Art. 24. *A pressure transmitted directly to any distance is not altered.* Thus let a weight *P*, fig. 3, be supported by a hand at *Q*, by means of a string *QP*: the force exerted by the hand is equal to the weight of *P*, whatever be the length of the string. For, instead of the weight at *P*, let a hand, as at *R*, exert a force equal to the weight; then this force will be supported as before, because the force of the hand *R*, produces the same effect as was produced by the weight *P*. But in this case it is clear that the forces exerted by *Q* and *R* must be equal, whatever be the length of the string, because they are like forces, and act upon the string in the same manner at

the two points. Hence the force exerted by Q is equal to the force exerted by R, that is, to the weight P. And thus the force which Q exerts in supporting P is the same, whatever be the length of the string.

Art. 27. *A pressure transmitted by a cord, round a peg or a pully, is transmitted without augmentation or diminution.*

In fig. 4, let a weight P be supported by a hand at Q acting by a string QOP round a fixed peg or pully O. Take $OR = OQ$, and let a hand act at R, with a force equal to the weight P: then, the force of R will still be supported by Q; the difference of the lengths OP, OR making no difference, by Art. 24. But the forces of Q and R, which thus balance each other, must be equal, because they are like forces, and act on the string QOR in exactly the same manner; the only difference being, that one of them is in a vertical direction, and the other not so; which difference cannot disturb their equilibrium, since neither of them depends at all upon gravity. Hence the force exerted by Q is always equal to that exerted by R, and therefore to the weight P: it is therefore the same whatever be the length or direction of the string OQ.

Art. 38, Axioms I, II, III may be illustrated by figures 5 and 6.

Art. 41. A cylinder BD, fig. 7, will produce the same effect on a lever CB as if it were collected at its middle point N. For this cylinder may be conceived as composed of pairs of equal small weights, as d and b, at equal distances from N, and each such pair will produce the same effect as if collected at N; and hence the whole cylinder will produce the same effect as if it were collected at N.

Art. 42. This proposition and proof may be illustrated by fig. 7.

Art. 43. This proposition may be illustrated by figures 8 and 9.

Art. 45. The levers of the first, second, and third kinds respectively, may be illustrated by figures 7, 8, and 9.

An oar may be considered as a lever of the second kind, the fulcrum being that point of the blade of the oar which is

for a moment stationary, going neither backwards nor forwards while the boat is impelled forwards: the *power* being the pull of the rower: and the *weight* being the pressure of the oar at the rowlock upon the side of the boat. The pull re-acts upon the boat and urges it backwards: the boat is impelled forwards by the excess of the weight over the power.

EXAMPLES of the *Straight Lever.*

Ex. 1. On a lever of the first kind, 3 feet long, a weight of 100 pounds is suspended at the extremity, and $2\frac{2}{3}$ inches from this end is placed a fulcrum; what weight at the other end will preserve the equilibrium?

In fig. 7, $MN = 36$ inches; $CM = 2\frac{2}{3}$ inches; $\therefore CN = 33\frac{1}{3}$ inches,

$$P : Q :: 33\tfrac{1}{3} : 2\tfrac{2}{3} :: 100 : 8;$$
$$\text{and } P = 100\text{ lbs.}; \therefore Q = 8\text{ lbs.}$$

Ex. 2. On a straight lever MO, fig. 10, let MC, CN, NN', $N'N''$, &c. be all equal; then if a weight Q be slid along the arm CO, what are the weights at M, which it will balance when at N, N', N'', &c.?

Q at N balances Q at M; Q at N' balances $2Q$ at M; Q at N'' balances $3Q$ at M, &c.

Hence, excluding the weight of the lever, the weight at M might be known from knowing the place of Q. We shall see hereafter how the weight of the lever itself may be taken into account.

If $CM = CN$, the weights at M and N are equal, and one of them may be used to measure the other. This is the case in the common balance, but when the arms are unequal, it is called a *false balance.*

Ex. 3. In a false balance, to find the true weight of the substance weighed.

Let CM, CN, fig. 10, be unequal, and let x be the weight to be determined. Let x at N be balanced by a ounces at M, and let x at M be balanced by b ounces at N. Therefore,

$$x : a :: CM : CN,$$
$$x : b :: CN : CM;$$

$$\therefore x^2 : ab :: 1 : 1,$$

$$x^2 = ab, \text{ and } a : x :: x : b;$$

$\therefore x$ is a mean proportional between a and b, the apparent weights in opposite scales.

Ex. 4. When a weight is supported on a lever at two points, to compare the pressures supported at the two points.

Let a weight R be supported on a lever MN, fig. 11, by forces P, Q. The same force is exerted at M as if N were a fulcrum: hence,

$$P : R :: NC : MN.$$

$$\text{So } R : Q :: NM : MC;$$

$$\therefore P : Q :: NC : MC.$$

Or, the pressures supported are inversely as the distances from the weight.

Art. 46. This proposition may be illustrated by fig. 12.

Art. 47. This proposition may be illustrated by fig. 13.

EXAMPLES of the *Bent Lever*.

Ex. 1. Fig. 13. P is 99 pounds, Q 100 pounds; $CA = 9$, $CB = 5$, and the angle $CAP = 30^0$; to find the angle CBN, that there may be an equilibrium;

$$P . CA . \sin A = Q . CB . \sin B;$$

$$\therefore \sin B = \frac{P . CA}{Q . CB} . \sin A = \frac{99 . 9}{100 . 5} . \frac{1}{2}$$

$$= .891 = \sin 62^0,$$

as appears by the tables of sines.

Ex. 2. In a straight lever AB, fig. 15, acted on by weights P, Q; if there be an equilibrium when it is horizontal, there will be an equilibrium in every position.

Let AB be any position of the lever; MCN a horizontal line. And if there be an equilibrium in the horizontal position

$$P : Q :: CB : CA.$$

But, by similar triangles, $CB : CA :: CN : CM$; therefore

$$P : Q :: CN : CM;$$

and therefore the equilibrium subsists.

Ex. 3. In a bent lever ACB, (without weight) fig. 16, having given the lengths of the arms, the angle which they make, and the weights P, Q, appended to them; to find the position in which it will rest.

Draw MCN horizontal, meeting PA, QB, in M, N. Let $CA = a$, $CB = b$, $ACB = \omega$; and $ACM = \theta$, which is to be found. Therefore $BCN = \pi - \omega - \theta$, π being two right angles.

$$P \, . \, CA \, . \cos ACM = Q \, . \, CB \, . \cos BCN,$$

$$Pa \cos\theta = Qb \cos(\pi - \omega - \theta) = -Qb \cos(\omega + \theta)$$

$$= -Qb \{\cos\omega \cos\theta - \sin\omega \sin\theta\},$$

$$(Pa + Qb \cos\omega)\cos\theta = Qb \sin\omega \sin\theta;$$

$$\therefore \tan\theta = \frac{Pa + Qb\cos\omega}{Qb\sin\omega}.$$

Ex. 4. In the same case, having given P, to find Q, such that the arm CA may be horizontal.

In this case, $\theta = 0$;

$$\therefore Pa + Qb\cos\omega = 0; \; Q = -\frac{Pa}{b\cos\omega}.$$

The problem will not be possible, except ω be greater than a right angle, in which case $\cos\omega$ is negative, and Q is positive.

Ex. 5. To find the force requisite to draw a carriage-wheel over an obstacle, supposing the weight of the carriage collected at the axis of the wheel.

Let A, fig. 17, be the axis of the wheel, CD the obstacle. Then if the wheel turn over the obstacle, it must turn round the point C; and the force which moves it being supposed to act in the line AP, and the weight in the vertical line AE, the wheel will be a lever such as

that referred to in Cor. 4, Art. 47. Hence, in order that P may balance the weight Q,

$$P : Q :: CN : CM :: \sin CAE : \sin CAP,$$

$$P = Q \cdot \frac{\sin CAE}{\sin CAP}.$$

Hence, P is least when $\sin CAP$ is greatest, or when CAP is a right angle. In this case, $P = Q \sin CAE$.

If the wheel be made larger, the obstacle being the same, the versed sine NE, or CD, remains the same; and the radius being increased, the angle CAE is diminished. Hence, *cæteris paribus*, P is diminished, and the larger the wheel, the smaller is the force requisite.

Art. 48. This proposition may be illustrated by fig. 18.

In fig. 18, P, Q, tend to turn the lever one way, and P', Q', tend to turn it the other way. By Art. 48 of the text there will be an equilibrium, if

$$P . CM + Q . CN + \&c. - P' . CM' - Q' . CN' - \&c. = 0;$$

that is, if

$$P . CM + Q . CN + \&c. = P' . CM' + Q' . CN' + \&c.$$

And conversely, if there be an equilibrium

$$P . CM + Q . CN + \&c. = P' . CM' + Q' . CN' + \&c.$$

Examples of *several Forces on a Lever.*

Ex. 1. In fig. 19, let P, Q, P', Q', be weights of 3, 5, 7, 9 pounds respectively, and MN, NM', $M'N'$, equal distances of one foot: to find the point on which the weights will balance.

Let $MN = NM' = M'N' = a$, and $MC = x$;

$$\therefore NC = x - a, \quad CM' = 2a - x, \quad CN' = 3a - x;$$

and therefore, by last Corollary,

$$3x + 5(x - a) = 7(2a - x) + 9(3a - x);$$

$$\therefore x = \frac{5a + 7 . 2a + 9 . 3a}{3 + 5 + 7 + 9} = \frac{46a}{24} = \frac{23a}{12};$$

$\therefore x = 23$ inches, and C is one inch from M'.

Ex. 2. To shew how the steelyard must be graduated.

The steelyard is a lever AB, fig. 20, which is moveable about a center C, and on which substances to be weighed are suspended from the extremity B, as at Q. A known weight P, moveable along the arm CA, is placed at such a distance from C as to balance the body Q: then from the place of A we may know the weight Q: and, if at different points of CA we place figures to represent the corresponding weights of Q, the arm CA is *graduated*.

The lever is now supposed to have weight, and the arm CA being longer and consequently heavier than the other, will preponderate. Suppose, that when Q and P are removed, a weight equal to P, placed at D, would keep the beam horizontal. If we then take $CO = CD$, it appears that the whole beam AB produces the same effect as a weight P placed at O, for either of the two would balance P, placed at D. Now let P and Q balance at B and M: therefore, Q balances P at M, together with the beam; that is, Q balances P at M, together with a weight which produces the same effect as P at O does. Hence,

$$Q \cdot CB = P \cdot CM + P \cdot CO = P \cdot CM + P \cdot CD$$
$$= P \cdot DM.$$

Hence, if we make DE, DF, DG, &c. equal to CB, $2CB$, CB, &c. we shall have, when P is at E, at F, at G, &c.

$$Q = P,\ Q = 2P,\ Q = 3P,\ \&c.$$

And therefore, the beam is graduated by taking such equal distances from the point D, and numbering the points thus found 1, 2, 3, &c.

Art. 54. This proposition and proof may be illustrated by fig. 21.

Art. 55. This proposition and proof may be illustrated by fig. 22; in which instead of the letters C, D, r, in the text, we have r, x, y.

Art. 56. Cor. 1. If three forces which keep a point in equilibrium act in the directions of three lines forming a triangle, they are proportional in magnitude to the sides of the triangle.

Let ABC, fig. 23, be the triangle. If one of the forces, as that in direction BC, be not represented in magnitude by BC, let it be represented by some other line, as BC': then, by Art. 56, the two forces AB, BC' are equivalent to AC'; and therefore cannot balance a force in direction CA. Therefore the forces cannot keep a point in equilibrium; which is contrary to the supposition.

Cor. 2. If three forces keep a body in equilibrium, and three lines be drawn making with the directions of the forces three equal angles towards the same parts; these three lines will form a triangle whose sides will represent the three forces respectively.

Let AB, BC, CA, fig. 24, and 25, be the directions of the forces; DM, EN, FO three lines such that the angles ADM, BEN, CFO are equal; these lines, produced if necessary, form a triangle abc. In the triangles aEM, ADM, the angle aME equals AMD, and by supposition aEM equals ADM; hence the remaining angle MaE or bac equals MAD or BAC; similarly, the angle abc equals ABC, and bca equals BCA. Hence the triangles abc, ABC are equiangular, and therefore

$ab : bc :: AB : BC$

:: force in direction AB : force in direction BC, by Cor. 1. And similarly of ca.

If therefore ab represent the force in direction AB; bc, ca will represent the forces in directions BC, CA.

Cor. 3. If the angle between two given forces be diminished, the resultant is increased.

Let two forces Ap, Aq, fig. 26, act at the angle pAq; pr being equal and parallel to Aq, Ar is the resultant.

Let Ap, AQ, the same forces, act at the angle pAQ; pR being equal and parallel to AQ, AR is the resultant.

pR is equal to pr, and if the angle $pAQ < pAq$, we have $Apr > Apr$ and therefore $AR > Ar$. (Euc. xxiv. 1.)

Art. 62. Examples of the *Composition of Forces.*

Ex. 1. Suppose a boat fastened to a fixed point by a rope, and acted on at the same time by the wind and the current. Then the direction of the rope will indicate the direction of the resultant of these actions.

In fig. 1, let BQ, BR, the directions of two forces which act at B, be at right angles, and let the forces exerted be 48 pounds, and 20 pounds: to find the magnitude and direction of the resultant.

If we make BRS a right angle, and $BR = 48$, $RS = 20$, BS will be the resultant. And $BS^2 = 48^2 + 20^2 = 2704$; $\therefore BS = 52$, and the resulting force is 52 pounds.

Also to find the angle SBR, we have $\sin SBR = \frac{20}{52} = \frac{5}{13}$; $\therefore SBR = 22^0\,37'$ nearly.

Ex. 2. We have many examples of the resolution of forces, in cases where the force exerted being resolved into two, one of them is somehow lost or counteracted, and the remaining part only is effective. Thus, if we would drag an object along the ground by a rope attached to it, if we suppose this rope to be inclined to the horizon at an angle of 45^0, the force which we exert is effective only in part. If we thus exert a force of 17 pounds, this force is equivalent to two equal forces, one in a horizontal and one in a vertical direction. And if each of these be called x, we shall have

$$x^2 + x^2 = 17^2,\ x = \frac{17}{\sqrt{2}} = 12 \text{ nearly.}$$

Hence the force with which we draw the body horizontally is 12 pounds.

Art. 59. This proposition may be illustrated by fig. 27.

Cor. 1. In the case of four or more forces, it does not follow conversely, as in the case of three forces, that if they act in the direction of the sides of the polygon and are in equilibrium, they are proportional to the sides. For the directions of the sides may remain the same while their proportions

are altered. Thus, if we draw $D'E'$ parallel to DE, forces parallel to the sides of the polygon will keep a point at rest, if they be proportional to AB, BC, CD', $D'E'$, $E'A$, as well as if they be proportional to AB, BC, CD, DE, EA.

Cor. 2. To find the resultant of forces which are not in the same plane.

Let AB, AC, AD, fig. 28, be three forces not in the same plane. Let the planes BC, BD, CD, be drawn, and the planes DG, CG, BG, parallel to them, completing the parallelepiped, whose sides will be parallelograms. Join AF, DG; DF will be a parallelogram, as is evident; and by Art. 55,

$$AB, AC, \text{ are equivalent to } AF;$$

$$\therefore AB, AC, AD, \text{ are equivalent to } AF, AD; \text{ that is, to } AG.$$

Hence, if the edges of a parallelepiped, drawn from the same point, represent the components, the diagonal will represent the resultant.

Cor. 1. If $ABEG$ be any four-sided figure, not all in the same plane, and if AB, BE, EG, represent three forces, AG will represent their resultant.

Cor. 2. If four forces acting upon a point, be represented by the sides of *any* four-sided figure, taken in order, they will keep the point at rest.

Cor. 3. If any number of forces be represented by sides, taken in order, of a polygon, which is not in the same plane, their resultant will be represented by the line which completes the polygon.

Cor. 4. If any number of forces be represented by all the sides, taken in order, of any polygon, they will keep a point at rest.

The three last Corollaries are proved from this Article, as those of Art. 30. are from Art. 30.

From the preceding principles, we may find the conditions under which a point will be kept in equilibrium, as will appear in the following Problems.

PROB. I. Fig. 29. A, B *are two points in the same horizontal line, and* AC, BC, *two strings from which, at the knot* C, *the weight* W *hangs: to find the forces exerted by the strings* CA, CB.

The point at which the equilibrium is produced is, in this case, the point C; and the forces which produce it are the forces of the strings CA, CB, and the weight W acting by the string CW. From any point d in the vertical line WC produced, draw db, da, parallel to CA, CB. In order to support the weight W the resultant of the forces of the strings CA, CB, must be in the direction Cd and must be equal to the weight W. The forces must therefore be as Ca, Cb, and their resultant will then be as CD by Art. 55. Hence if Cd represent the weight W, we have the forces of the strings represented by Ca, Cb. Or, if P, Q represent the forces of the strings CA, CB; we have

$$\frac{P}{W} = \frac{Ca}{Cd} = \frac{\sin Cda}{\sin Cad} = \frac{\sin dCb}{\sin aCb} = \frac{\sin DCB}{\sin ACB};$$

$$\text{similarly, } \frac{Q}{W} = \frac{\sin DCA}{\sin ACB};$$

whence P and Q are known.

COR. 1. The forces of the strings measure their *tensions*, (see Art. 23.) and these again are measured by the pressures exerted on the immoveable points A, B. But if instead of supposing the strings fixed at the points A, B, we suppose them to pass over those points, or over pullies placed there, and to have appended to them weights equal to the forces P, Q; these weights will be just supported, that is, there will be an equilibrium. See Art. 27.

PROB. II. Fig. 30. *Two strings* CAP *and* CBQ *pass over pullies* A *and* B, *in the same horizontal line, and support a weight* W *by means of equal weights* P *and* Q *suspended at their other extremities: to find the position of the point* C.

Draw lines as in the preceding problem, and let Cd meet AB in E: then by the last corollary the weights W, P, Q will be

as Cd, Ca, Cb; and since the weights P and Q are equal, $Ca = Cb = ad$; $\therefore$ $\angle aCd = Cda = dCb$; $\therefore$ the triangles ACE, BCE are equal, and $AE = EB$. Hence E bisects AB, and C will be in the vertical line passing through E.

Join ab meeting cd in e; aec, AEC are right angles.

And $\dfrac{P}{W} = \dfrac{Ca}{Cd} = \dfrac{Ca}{2Ce} = \dfrac{CA}{2CE}$ by similar triangles.

Let $AE = EB = a$, $EC = x$; $\therefore$ $CA = (a^2 + x^2)^{\frac{1}{2}}$,

$$\frac{P}{W} = \frac{(a^2 + x^2)^{\frac{1}{2}}}{2x};$$

$$\therefore \frac{4P^2}{W^2} x^2 = a^2 + x^2; \quad \frac{4P^2 - W^2}{W^2} x^2 = a^2;$$

$$x = \frac{Wa}{(4P^2 - W^2)^{\frac{1}{2}}}.$$

Whence the position of C is known.

COR. 1. In order that x may be possible, we must have the quantity under the radical sign positive, and therefore

$$W^2 < 4P^2,$$

$$\text{or } W < 2P:$$

if W be equal to or greater than $2P$, it will descend, drawing up both the weights, and will never find a place where it will rest.

COR. 2. In order that the string ACB may be drawn so as to be in the horizontal line, we must have $x = 0$,

$$\text{or } \frac{Wa}{(4P^2 - W^2)^{\frac{1}{2}}} = 0;$$

which cannot be, except either W be indefinitely small or P indefinitely great. That is, no weights P, Q, however great, can draw up a weight W, so that the string ACB shall be a horizontal straight line. If ACB, instead of being a line without weight loaded with a weight at its middle, be a cord of which each part has weight, the same will be true.

PROB. III. Fig. 31. P, Q, *support* W *as in the last Problem, the values of* P, Q, *and the positions of the pullies* A, B, *being any whatever; to find the position of equilibrium of* C.

As before, let Cd be vertical, and da parallel to BC. Then P, Q, W, are as Ca, ad, dC. Hence, in the triangle Cad, we have the proportions of three sides given, to find the angles aCd, Cda.

$$\begin{aligned} \text{Also } BAC &= BAP - CAP = BAP - aCd, \\ ABC &= ABQ - CBQ = ABQ - bCd \\ &= ABQ - Cda. \end{aligned}$$

Hence, knowing the position of the points A, B, and therefore the angles BAP, ABQ, we know the angles BAC, ABC; and hence knowing the side AB, we may solve the triangle ABC, and calculate the position of C. (See Art. 62.)

PROB. IV. *A string* ACDEB, fig. 32, *of which the extremities* A, B, *are fixed, is kept in a given position by weights* P, Q, R, *suspended at knots* C, D, E; *to compare the weights* P, Q, R.

Let the sides of the polygon AC, CD, DE, EB make with the horizontal line angles β, γ, δ, ϵ. Then it is easily seen that if AC be produced to c, $DCc = \beta - \gamma$. Similarly, $EDd = \gamma - \delta$, &c.

The point C is kept at rest by three forces; viz. the weight P, the *tension* of CA, and the *tension* of CD: let the latter be called C, and we shall have, by Art. 58,

$$\begin{aligned} \frac{P}{C} &= \frac{\sin ACD}{\sin ACP} = \frac{\sin DCc}{\sin ACx} \\ &= \frac{\sin(\beta - \gamma)}{\cos\beta} = \frac{\sin\beta\cos\gamma - \cos\beta\sin\gamma}{\cos\beta} \\ &= \cos\gamma\,(\tan\beta - \tan\gamma). \end{aligned}$$

Also the point D is kept as rest by three forces; the weight Q, the tension of DE, and the tension of CD at D; and the last is the same as C, the tension of CD at C, because the string must be kept at rest by equal and opposite forces, and therefore must exert equal and opposite tensions at its two extremities.

$$\text{Hence, } \frac{Q}{C} = \frac{\sin(\gamma - \delta)}{\cos\delta}$$

$$= \frac{\sin\gamma\cos\delta - \cos\gamma\sin\delta}{\cos\delta}$$

$$= \cos\gamma(\tan\gamma - \tan\delta).$$

Hence, we have

$$\frac{Q}{P} = \frac{\tan\gamma - \tan\delta}{\tan\beta - \tan\gamma}.$$

Similarly, we shall have the proportions of the forces at the other angles.

Hence, P, Q, R, are proportional to the differences of the tangents of the angles which the supporting strings make with the horizon.

If one of the strings, as EB, have that end higher which is farther from the origin A, the corresponding angle ϵ is to be taken negative, and we shall still have, ($-\tan\epsilon$ being a positive quantity,)

$$\frac{Q}{R} = \frac{\tan\gamma - \tan\delta}{\tan\delta - \tan\epsilon}.$$

Cor. If the forces P, Q, R, instead of being parallel, were to make any angles with each other, we should be able to compare them by the application of Art. 58.

A cord kept in equilibrium in such a manner is called a *Funicular polygon*.

Prob. V. Fig. 33. *A cord* PAQ, *which passes round a fixed point* A, *is drawn in different directions by forces* P, Q; *to find the pressure upon the point* A.

In the first place, the forces P, Q must necessarily be equal, for, as the string passes freely round A, the forces will balance each other in the same manner as if they acted at the two ends of a string which was in a straight line, and therefore they will be equal. Now if we suppose A, instead of being immoveable, to be retained in its place by a force, as AR, this force must manifestly, with the forces in AP and AQ, produce equilibrium

at the point A. Hence, if we produce RA to any point r, and draw rp parallel to AQ; Ap, pr, rA will, by Art. 57, be proportional to the forces in AP, AQ, and AR. Also it has been shewn that the forces in AP, AQ are equal, and therefore Ap, pr are equal, and the angle $rAp = Arp = rAQ$. Hence Ar bisects the angle PAQ, and if po be perpendicular to Ar,

$$Ar = 2Ao = 2Ap \cos pAr = 2Ap \cos \tfrac{1}{2} PAQ.$$

Hence, if we put the forces P, Q, each $= P$, and the force in $AR = R$; also $\angle PAQ = A$,

$$\frac{R}{P} = \frac{Ar}{Ap} = \frac{2Ap\cos\frac{1}{2}A}{Ap} = 2\cos\tfrac{1}{2}A;\ \therefore\ R = 2P\cos\tfrac{1}{2}A;$$

and R, the force which would keep A at rest, is evidently equal to the pressure upon that point produced by the cord PAQ: hence we have the pressure upon $A = 2P\cos\frac{1}{2}A$.

Cor. 1. If A, instead of a point, be a pully round which the cord passes, the pressure on the pully will be the pressure at the center of the pully. For in this case, fig. 34, the strings aP, bQ, touch the circle abd of the pully, and would if produced meet in the line CA which passes through the center, and would make equal angles with it. Hence the resultant of the tensions in aP, bQ passes through the center, and is, as before, equal to $2P\cos\frac{1}{2}A$.

Cor. 2. If a string pass over any number of fixed points $ABCD$, and be kept at rest by forces or weights P, Q, drawing it in opposite directions, these forces or weights must be equal. And the pressure upon any one of the points, as B, will be $2P\cos\frac{1}{2}ABC$.

Prob. VI. Fig. 35. *A given weight* W *is supported by two props* AC, BC *upon a horizontal plane* AB. *To find the pressure upon each prop, their lengths and the distance at which they stand being given.*

If we take Cd in the vertical line CD to represent the weight of the body, and draw da parallel to BC, Ca, ad will represent the pressures (Art. 57.); but, to prepare the student

for the solution of succeeding problems, we shall obtain them by a different method.

Let the *re-actions* of the props in the direction AC, BC, be P, Q, (see Art. 33.) Let P be resolved in the horizontal and vertical directions AD, DC. Then

$$\frac{\text{horizontal part of } P}{P} = \frac{AD}{AC} = \cos A;$$

$$\frac{\text{vertical part of } P}{P} = \frac{DC}{AC} = \sin A;$$

and similarly for Q.

$\therefore$ horizontal force of AC at $C = P.\cos A$; of $BC = Q.\cos B$;
vertical force of AC at $C = P.\sin A$; of $BA = Q.\sin B$.

And, since these forces support the weight, the horizontal parts must counteract each other, and the vertical parts must together $= W$;

$$\therefore\ P\cos A = Q\cos B;$$

$$P\sin A + Q\sin B = W.$$

By the first, $Q = \dfrac{P\cos A}{\cos B}$; hence, by the second,

$$P\sin A + \frac{P\cos A}{\cos B}\sin B = W;$$

$$\therefore\ P(\sin A\cos B + \cos A.\sin B) = W\cos B;$$

$$\text{or } P.\sin(A+B) = W\cos B;$$

$$\text{or } P.\sin C = W\cos B;$$

$$\text{and } P = \frac{W\cos B}{\sin C}.$$

$$\text{Similarly, } Q = \frac{W\cos A}{\sin C}.$$

Wherefore, as we can express $\cos A$, $\cos B$, $\sin C$, in terms of AC, BC, AB, we can thus obtain the forces or re-actions P, Q. And the pressures upon the props are equal to these the re-actions which the props exert.

Cor. 1. If we make AC, BC, AB equal to a, b, c, respectively, we shall have

$$\cos B = \frac{b^2 + c^2 - a^2}{2bc}, \text{ (by Trigonometry,)}$$

$$\sin C = \frac{c\sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}}{2ab};$$

$$\therefore P = \frac{Wa \cdot (b^2 + c^2 - a^2)}{c\sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}},$$

$$\text{and } Q = \frac{Wb \cdot (a^2 + c^2 - b^2)}{c\sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}}.$$

Cor. 2. The props exert upon the plane at A and B pressures equal to those which are exerted on their upper extremities: these pressures at A and B may be resolved in directions perpendicular and parallel to the plane.

The parts perpendicular to the plane will be

$$\frac{W \cdot \cos B \cdot \sin A}{\sin C} \text{ at } A, \text{ and } \frac{W \cdot \cos A \cdot \sin B}{\sin C} \text{ at } B,$$

and these are counteracted by the re-action of the plane.

The parts parallel to the plane will be

$$\frac{W \cdot \cos B \cdot \cos A}{\sin C} \text{ at } A, \text{ and } \frac{W \cdot \cos A \cdot \cos B}{\sin C} \text{ at } B;$$

and these, if not counteracted, will make the props slide in opposite directions from A and B along the horizontal plane. They may be counteracted by immoveable obstacles placed behind the props at A and B. They will sometimes be counteracted by the friction of the plane.

Prob. VII. Fig. 36. *A Weight* W *is supported by three props* AW, BW, CW, *upon a horizontal plane* ABC. *To find the pressure on each: the lengths of the props and the distances at which they stand being given.*

Draw WO perpendicular to the horizontal plane, join AO, and produce it to meet BC in K, and join WK.

The pressures of the three props in their own directions together support the weight, and therefore produce a pressure in the vertical direction OW; also the pressure of AW will not be altered if we substitute for the pressures of BW, CW a pressure equivalent to them both; and this equivalent pressure must, along with AW, produce a vertical pressure in OW; hence it must be in the plane AWO, and therefore in the line KW, for it must manifestly be in the plane BWC; hence the weight W may be supposed to be supported by two props AW, KW, and the pressure on AW found by the last problem. Let as before P, Q, R, represent the pressures of the props AW, BW, CW; then

$$P = W \cdot \frac{\cos AKW}{\sin AWK}; \text{ and similarly,}$$

$$Q = W \cdot \frac{\cos BLW}{\sin BWL},$$

$$R = W \cdot \frac{\cos CMW}{\sin CWM}.$$

COR. 1. Since WO is perpendicular to AK, we have

$$\cos AKW = \frac{OK}{KW},$$

$$\sin AWK = \frac{AK}{KW} \cdot \sin KAW = \frac{AK}{KW} \cdot \frac{OW}{AW};$$

$$\text{hence, } P = W \cdot \frac{OK}{KW} \cdot \frac{KW}{AK} \cdot \frac{AW}{OW}$$

$$= W \cdot \frac{OK}{AK} \cdot \frac{AW}{OW}; \text{ and similarly,}$$

$$Q = W \cdot \frac{OL}{BL} \cdot \frac{BW}{OW};$$

$$R = W \cdot \frac{OM}{CM} \cdot \frac{CW}{OW}.$$

Cor. 2. If we draw AD, OE perpendicular to BC, we shall have

$$\frac{OK}{AK} = \frac{OE}{AD}; \text{ and hence } P = W.\frac{OE}{AD}.\frac{AW}{OW}$$

$$= W.\frac{OE}{OW}.\frac{AW}{AD}$$

$$= W.\frac{1}{\tan OEW}.\frac{AW}{AD};$$

and similar expressions may be obtained for Q and R.

Cor. 3. It is easily shewn that WE is perpendicular to BC; hence OEW measures the inclination of the planes CBA, CBW. Hence if a sphere with radius = 1, and center C, cut the pyramid, and make a spherical triangle efg, the angle e will be equal to the angle OEW. And if the angles made by the lines CA, CB, CW are known, the sides ef, fg, ge are known, and e may be found.

Cor. 4. The horizontal pressures which are to be resisted by obstacles at the lower ends A, B, C, are in the directions OA, OB, OC, and are equal to

$$P.\frac{OA}{AW} = W.\frac{OK}{AK}.\frac{OA}{OW};$$

$$Q.\frac{OB}{BW} = W.\frac{OL}{BL}.\frac{OB}{OW};$$

$$R.\frac{OC}{CW} = W.\frac{OM}{CM}.\frac{OC}{OW}.$$

Cor. 5. If the point O fall without the triangle ABC, the weight W cannot be supported.

CHAPTER VIII.

THE MECHANICAL POWERS.

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101. *MACHINES*, or, as they are called in their simplest state, *the Mechanical Powers*, are contrivances to enable a smaller force to keep at rest, or to put in motion a larger weight, or to overcome a greater resistance. We shall at present only consider the case where *equilibrium* is produced; for, knowing the force which would, by means of any machine, just support a weight, it is manifest that a larger force would raise it.

In these cases, as in the case of the Lever, (see Art. 44) the force applied is called *the Power*, and the resistance overcome is called *the Weight*, and is measured by a weight to which it is equivalent.

The mechanical powers may be reduced to THE LEVER, THE WHEEL AND AXLE, THE TOOTHED WHEEL, THE PULLY, THE INCLINED PLANE, THE WEDGE, AND THE SCREW*.

The four first are, in the state of equilibrium, reducible to the Lever. The Screw may be reduced to the Inclined Plane, as may the Wedge. The way in which the latter is considered is not immediately applicable to it in its common use: instead of being kept at rest by pressure, and put in motion by excess of pressure, as is supposed in our reasonings, it is practically kept at rest by friction, and put in motion by impact.

* A more complete classification of simple machines is given in the *Mechanics of Engineering*, Chapter I, to this effect.—*Machines* consist of *Pieces* which transmit force and motion, and which may be constrained in their motion by turning about a fixed *Axis*, or by connexion with one another. The case in which a Piece is moveable about an Axis includes the LEVER, and the WHEEL AND AXLE. Pieces may act on each other by *attachment at a point*, by *contact of sliding surfaces*, or by *contact of wrapping bands*. Action with contact of sliding surfaces includes, as simple examples, the INCLINED PLANE, the WEDGE, the CAM, TOOTHED WHEELS, and the SCREW. Action with contact of wrapping bands includes, as simple cases, PULLIES. When two pieces are connected by a rigid Piece attached to each at a point, this piece is a LINK; as for example AB, in Fig. 59[a].

SECTION I.

MECHANICAL POWERS REDUCIBLE TO THE LEVER.

1. *The Lever.*

This instrument has already been considered in Chap. I.

2. *The Wheel and Axle.*

102. *The wheel and axle* consists of a cylinder or *axle* AB, fig. 37, and a concentric circle or *wheel* EF, joined together, so that the whole is moveable about the axis of the cylinder: the weight W is attached to a cord NW, and will manifestly be raised or lowered as the wheel and axle are turned one way or the other. It is supported by a force applied at the circumference of the wheel EF, either by another weight P acting by means of a string wrapped the contrary way to that at N, or by some other force as P', acting at a point M' in the circumference of the wheel.

PROP. *In the wheel and axle the power is to the weight as the radius of the axle to the radius of the wheel*.*

Let fig. 38 be a representation of the machine referred to the plane EF, which is perpendicular to the axis. It is evident that the equilibrium will continue to subsist, if we suppose P and W, retaining their distances from the axis, to act in this plane. Let them act in the vertical lines MP and NW, and let MCN be a horizontal line through the center. Hence, considering MCN as a lever,

$$P : W :: CN : CM \text{ by Art. 15.}$$
$$:: \text{rad. of axle} : \text{rad. of wheel.}$$

It is obvious, that in the state of equilibrium this is the same machine with the lever. When they are put in motion, the two machines differ. In the wheel and axle the weight W ascends or descends in a vertical line: in the lever it describes a circular arc.

COR. 1. The power may act by means of a bar CM', and the wheel may be removed; this is the case in the *capstan* and *windlass*.

* In this and the following Propositions of this Chapter, the machines are supposed to be in equilibrium.

Cor. 2. If the direction of the power be not perpendicular to CM, we must draw a perpendicular upon it from C, and the proportion will be

$$P : W :: \text{rad. of axle} : \text{per. on dir}^{\text{n}}\text{. of power.}$$

3. *Toothed Wheels.*

103. If two circles W, B, fig. 39, moveable about their centers, have their circumferences indented or cut into equal *teeth*, all the way round, and be so placed that their edges touch, as at Q, the prominences of one of them at that part lying in the hollows of the other; then if one of them, as A, be turned round by any means, the other will be turned round also. Such circles are called *Toothed Wheels.*

If we suppose the two circles in fig. 39, to be in the same plane, and if, one of them A being turned by a power P acting on a winch CE, the other raise a weight W by means of an axle DF, we shall have the proportion of P and W by the following proposition.

Prop. *In toothed wheels, the moment of* P *about the center of the first wheel is to the moment of* W *about the center of the second wheel, as the perpendiculars from the centers of the wheels upon the line of direction of their mutual action.*

The edges of the teeth which act upon one another are conceived to be perfectly smooth; that is, they are supposed by their pressure to exert only a force perpendicular to their surface. All the effect produced to resist motion along a surface is supposed to arise from a defect of smoothness. If the pressure exerted at the point of contact were not perpendicular to the surface pressed, this pressure might be resolved into two forces, one perpendicular to the tangent, and the other in the direction of the tangent, and the latter force is understood to arise from friction, &c. and is at present left out of consideration.

Let the wheel A act upon the wheel B at Q; the action there exerted will be perpendicular to the surfaces which are in contact at that point: and the action of A on B, and the reaction of B on A will be equal and opposite: let this action be a pressure Q in the direction MQN. Then the force Q acting

on the wheel B supports the weight W, and the re-action opposite to Q is supported by the power P. Hence, if CM, DN, be perpendiculars on MQN, we shall have, by Art. 47,

$$P : Q :: CM : CE,$$

$$Q : W :: DF : DN;$$

$$\therefore P : W :: CM \,.\, DF : DN \,.\, CE.$$

Hence multiplying the first and third terms by CE, and the second and fourth by DF, we shall have

$$P \,.\, CE : W \,.\, DF :: CM : DN,$$

or mom. of P : mom. of W :: CM : DN.

Cor. 1. If CD meet MN in O, we have, by similar triangles,

$$CM : DN :: CO : DO;$$

$\therefore$ mom. of P : mom. of W :: CO : DO.

Cor. 2. If the form of the teeth be such, that the point O is fixed while the wheels revolve, the force continues the same during the motion.

This is the case when the form of the teeth is the involute of a circle.

Cor. 3. If the teeth be small in comparison with the radii of the wheels, Q will nearly coincide with O; and CO, DO will be very nearly the radii of the wheels measured to the point at which the contact takes place. Hence

mom. of P : mom. of W :: rad. of A : rad. of B.

Cor. 4. In order that the two wheels may work during a whole revolution, the intervals of their teeth must be equal; hence the numbers of teeth in each wheel will be as the circumferences, and therefore as the radii: hence

mom. of P : mom. of W :: number of teeth of A : number of teeth of B.

Cor. 5. The case in the figure is a combination of a winch, two toothed wheels, and an axle. If we suppose the

radius of the axle DF and the winch CE to be equal, the whole of the mechanical advantage will be owing to the toothed wheels. In this case, we have

$$P : W :: CO : DO.$$

When the number of teeth in A is very small, A is called a *Pinion*, and its teeth are called *Leaves.*

The teeth in which those of the wheel A work may be distributed along the edge of a straight bar instead of the circumference of a circle, the bar being restrained to move in the direction of its length.

Wheels are sometimes turned by simple contact with each other; sometimes by the intervention of cords, straps, or chains, passing over them; and in these cases the minute protuberances of the surfaces, or whatever else may be the cause of friction, prevents their sliding on each other. And at the points of contact an action and re-action are exerted corresponding to those which are supposed in the Proposition.

4. *Pullies.* (1) *The single moveable Pully.*

104. A pully has already been mentioned, Art. 27, &c., as a means of changing the direction of part of the cord by which force is exerted; it is a small wheel which is moveable about its axis, and along part of the circumference of which the cord passes. So long as its axis is immoveable, it can produce only a change of direction; but when its axis is fixed in a *block* or *sheaf* which is moveable, it may produce a mechanical advantage.

PROP. *In the single moveable pully, the strings being parallel,*

$$P : W :: 1 : 2.$$

Let $CBAP$, fig. 40, be the cord passing round the pully AB; and let the force P act by this cord. By Art. 27, the tension of the string is the same throughout, and equal to the power P. Hence AB is supported by two equal and parallel forces in AP, BC; each equal to P; and hence, by Art. 25, the force W, which acts in the opposite direction upon AB, must be equal to their sum. Therefore $W = 2P$.

COR. 1. If the strings be not parallel, as KA, CB, fig. 41, let them be produced and meet in n; and join on, o being the center of the pully. Then oA and oB, drawn to the points where the string touches the pully, are equal, because the pully is circular. And on is common, and oAn, oBn, right angles. Hence onA, onB, are equal.

The strings AK, BC will produce the same effect as if they acted at n. And the forces or tensions exerted by them are equal, each being equal to P. Hence the resultant bisects the angle AnB, and is in the direction no: and since the forces of the strings support the weight, no must be opposite to the direction in which the weight acts; and therefore vertical.

Let a horizontal line meet the strings in p, q, and the vertical line nm in m. np, nq, will be equal, and may be taken to represent the tensions of the strings AK, BC. And (Art. 56,) np is equivalent to nm, mp, and nq to nm, mq. And of these, the parts mp, mq destroy each other; and hence the force acting upwards in $2nm$. Therefore,

$$P : W : np : 2nm$$

$$: \text{rad.} : 2\cos pnm.$$

If rad. $= 1$, and $pnm = \alpha$, $W = 2P\cos\alpha$.

If $\alpha = 0$, the strings are vertical, $\cos\alpha = 1$, and $W = 2P$, as before.

If $\alpha = 60^0$, or $AnB = 120^0$, $\cos\alpha = \frac{1}{2}$, and $W = P$.

COR. 2. When a weight is supported on a moveable pully, the two portions of the string make equal angles with the direction in which the weight acts.

COR. 3. We may deduce the relation of P to W, in fig. 40, by considering BA as a lever. For if we suppose the point B to be a fulcrum, and the weight W to be supported by a force P acting vertically at A; we have

$$P : W :: Bo : BA :: 1 : 2,$$

as before.

Hence, the pully, in the state of equilibrium, may be reduced to the lever.

(2) *First system of Pullies. Each Pully hanging by a separate String.*

105. The first system of pullies, fig. 42, is merely a repetition of the single moveable pully. The weight W is supported by the pully A_1; the string which passes round A_1 is supported by A_2; the string which passes round A_2 by A_3, and so on; and at the last string (which may pass over a fixed pully B) the power of P acts.

PROP. *In the first system of pullies, where all the strings are parallel, and the weights of the pullies inconsiderable,*

$$P : W :: 1 : 2^n;$$

n *being the number of moveable pullies.*

By last Article,

$$\text{tension of } A_1A_2 = \tfrac{1}{2} \text{ weight at } A_1 = \frac{W}{2},$$

$$\text{tension of } A_2A_3 = \tfrac{1}{2} \text{ weight at } A_2 = \tfrac{1}{2} \text{ tension of } A_1A_2 = \frac{W}{2^2},$$

$$\text{tension of } A_3B = \tfrac{1}{2} \text{ weight at } A_3 = \tfrac{1}{2} \text{ tension of } A_2A_3 = \frac{W}{2^3}.$$

And similarly, we should have, if A_n were the last of the moveable pullies,

$$\frac{W}{2^n} = \text{tension of } A_nB = \text{power } P,$$

for the tension of the string at which P acts is equal to P. Hence

$$W = 2^n P,$$

when n is the number of moveable pullies.

COR. 1. If A_1, A_2, A_3, &c. be the weights of the pullies, (including the blocks, &c.) respectively, we may consider each pully as a weight appended at that point: hence

$$\text{weight at } A_1 = W + A_1,$$

$$\text{tension of } A_1A_2 = \tfrac{1}{2} \text{ weight at } A_1 = \frac{W}{2} + \frac{A_1}{2};$$

$$\therefore \text{ weight at } A_2 = \frac{W}{2} + \frac{A_1}{2} + A_2;$$

$$\therefore \text{ tension of } A_2A_3 = \tfrac{1}{2} \text{ weight at } A_2 = \frac{W}{2^2} + \frac{A_1}{2^2} + \frac{A_2}{2};$$

$$\therefore \text{ weight at } A_3 = \frac{W}{2^2} + \frac{A_1}{2^2} + \frac{A_2}{2} + A_3;$$

$$\therefore \text{ tension of } A_3B = \tfrac{1}{2} \text{ weight at } A_3$$

$$= \frac{W}{2^3} + \frac{A_1}{2^3} + \frac{A_2}{2^2} + \frac{A_3}{2} = P;$$

and so on; and if there be n moveable pullies,

$$\frac{W}{2^n} + \frac{A_1}{2^n} + \frac{A_2}{2^{n-1}} \ldots\ldots + \frac{A_n}{2} = P;$$

$$\therefore W + A_1 + 2A_2 + \ldots\ldots + 2^{n-1}A_n = 2^nP.$$

Cor. 2. If the pullies be all equal, and each equal to A,

$$W + A(1 + 2 \ldots\ldots + 2^{n-1}) = 2^nP,$$

$$W + A(2^n - 1) = 2^nP,$$

$$W = 2^nP - A(2^n - 1).$$

Cor. 3. Hence the weight W is less as A is greater. If we have $2^nP = A(2^n - 1)$, W will $= 0$, and the power will only just support the pullies.

Cor. 4. If the strings be not parallel, we must compare the tension of each with that of the preceding by Cor. 1, to last Article.

(3) *Second system of Pullies. The same String passing round all the Pullies.*

106. This system, fig. 43, consists of two blocks; an upper one B_1B_2, and a lower one A_1A_2: each contains a certain number of pullies, and the string passes round them alternately. The weight is hung to the lower block, and the power acts at the loose extremity of the string.

PROP. *In the second system of pullies, if the strings be parallel,*

$$P : W :: 1 : n;$$

n *being the number of strings at the lower block.*

Since the same string passes round all the pullies, its tension will be every where the same, and equal to the power P. And n being the number of strings at the lower block, since each of them supports a weight P, they will altogether, supposing them parallel, support a weight nP: hence

$$W = nP.$$

COR. 1. If we consider the weight of the pullies, it is manifestly only requisite to add the weight of the lower block to W; hence if A be this block,

$$W + A = nP \; : \; W = nP - A.$$

COR. 2. If the strings be inclined to the vertical we must take the resolved part of the force. But the angle made with the vertical is generally so small that this correction may be omitted.

(4) *Third system of Pullies. Each String attached to the Weight.*

107. In this system, fig. 45, each string, as PA_1C_1, supports the weight, partly by its action at C_1, where it is attached, and partly by its pressure on the next string, as A_1A_2.

PROP. *In the third system of pullies, the strings being parallel, and the weight of the pullies inconsiderable,*

$$P : W :: 1 : 2^n - 1;$$

n *being the number of pullies.*

For, tension of $PA_1 = P$;

$$\therefore \text{ weight supported at } C_1 = P;$$

$$\text{tension of } A_1A_2 = \text{pressure on } A_1 = 2P;$$

$$\therefore \text{ weight supported at } C_2 = 2P;$$

$$\text{tension of } A_2A_3 = \text{pressure on } A_2 = 2^2P;$$

$$\therefore \text{ weight supported at } C_3 = 2^2P;$$

and so on.

Hence the whole weight W, which is the sum of all those supported at C_1, C_2, C_3, &c. is $W = P + 2P + 2^2P + \ldots\ldots = (1 + 2 + 2^2 \ldots\ldots + 2^{n-1}) . P$, if there be n pullies;

$$\therefore \; W = (2^n - 1) . P.$$

Cor. 1. If we consider the weights of the pullies, and call them A_1, A_2, $A_3 \ldots\ldots A_n$, we shall have

$$\text{tension of } PA_1 = \text{weight supported at } C_1 = P;$$

$\therefore$ pressure on $A_1 = 2P$;

$\therefore$ tension of $A_1 A_2$ = weight supported at $C_2 = 2P + A_1$;

$\therefore$ pressure on $A_2 = 2^2P + 2A_1$;

$\therefore$ tension of $A_2 A_3$ = weight supported at $C_3 = 2^2P + 2A + A_2$;

$\therefore$ pressure on $A_3 = 2^3P + 2^2A_1 + 2A_2 + A_3$;

and so on.

Hence, since the weight W (including the hook, &c. at C_1) is equal to the sum of all the weights supported, if n be the number of pullies;

$$\begin{aligned} W = &(1 + 2 + 2^2 \ldots\ldots + 2^{n-1}) P \\ &+ (1 + 2 + 2^2 \ldots\ldots + 2^{n-2}) A_1 \\ &+ (1 + 2 + 2^2 \ldots\ldots + 2^{n-3}) A_2 \\ &\ldots\ldots\ldots \\ &+ (1 + 2) A_{n-2} \\ &+ A_{n-1} \end{aligned}$$

$$= (2^n - 1) P + (2^{n-1} - 1) A_1 + (2^{n-2} - 1) A_2 \ldots\ldots + A_{n-1}.$$

Cor. 2. Hence, contrary to the other cases, W becomes greater by giving weight to the pullies. If we make $P = 0$, we may find the weight which will be supported by the pullies alone.

Cor. 3. If all the pullies be equal and each $= A$,

$$W = (2^n - 1) . P + (2^{n-1} - 1 + 2^{n-2} - 1 \dots\dots + 2 - 1) . A$$
$$= (2^n - 1) . P + [2^n - 2 - (n - 1)] . A$$
$$= (2^n - 1) . P + (2^n - n - 1) . A = (2^n - 1)(P + A) - nA.$$

Cor. 4. If the strings be not parallel, we must use Cor. 1, of the single moveable pully.

SECTION II.

MECHANICAL POWERS REDUCIBLE TO THE RESOLUTION OF FORCES.

5. *The Inclined Plane.*

108. An inclined plane, that is, a plane inclined to the horizon, is sometimes used as a mechanical power. Let a weight W, fig. 46, be supported on an inclined plane AC, (inclined to the horizon at an angle CAB) by a power P acting in the direction WK.

Prop. *In the inclined plane, if* WK *in the direction of the power and* WN *perpendicular to the plane, be intercepted by a vertical line* KN;

$$P : W :: WK : KN.$$

The effect of the plane AC on the weight W will be in a direction WR perpendicular to the plane; for the plane cannot produce any pressure on W in a direction parallel to AC or CA. (See Ax. xii.) Hence the weight will be supported in the same manner as if, instead of the plane AC, it were sustained by a string in the direction WR. It may therefore be considered as supported by a string WR, exerting a force equal to the re-action of the plane $(= R)$; a string WK exerting a force equal to the power P; and its weight $(= W)$ acting in the vertical direction WD.

Hence, since KN is parallel to WD, by Art. 56,

$$P : W :: WK : KN.$$

Similarly, $R : P :: WN : WK$.

Cor. 1. If the force act parallel to the horizon, in the direction WE, fig. 47, K coincides with E, E is a right angle, and

$$P : W :: WE : EN :: BC : AB, \text{ by similar triangles.}$$

In the same manner,

$$R : W :: WN : EN :: AC : AB.$$

Cor. 2. If the force act parallel to the plane, K coincides with C,

$$P : W :: WC : NC :: BC : AC,$$

$$R : W :: WN : NC :: AB : AC.$$

Cor. 3. If the force act perpendicular to the horizon, in the direction WZ, we must suppose the point K removed to an infinite distance, so that WK, NK, are infinite and equal. Hence

$$P = W, \ R = 0.$$

Cor. 4. If the force act so as to make an angle CWY below the plane equal to CWZ above it, take away from the right angles CWN, CWR, equal angles CWY, CWZ, and we have $YWN = ZWR$; and this $= YNW$ by parallels; therefore

$$YW = YN; \quad \therefore \ P = W.$$

$$\text{Also } YWC = YCW = CWZ; \ \therefore \ YW = YC,$$

$$\text{and } NC = 2\,YC = 2\,YN;$$

$$\therefore \ R : W :: WN : NY :: 2\,WN : NC :: 2AB : AC.$$

Cor. 5. If the force act in a direction WS, situate between WZ the perpendicular to the horizon, and WR the perpendicular to the plane, the point K will fall below N, as at M, and the weight, the power, and the re-action of the plane, are represented in magnitude and direction by NM, MW, WN. Hence the re-action of the plane is in the direction WM, and the body W is supported on the underside of the plane.

Cor. 6. In fig. 46, if we draw CV parallel to KW, we have

$$P : W :: KW : KN :: CV : CN.$$

Hence, W being given, P is least when CV is least, that is, when CV coincides with CW, or the force is in the direction of the plane.

Cor. 7. Let two weights W, W', fig. 48, support each other on two inclined planes AC, $A'C$, by means of a string which is parallel to the planes. Let P be the tension of the string, which will be the same on each, then, by Cor. 2,

$$P : W :: BC : AC,$$
$$W' : P :: A'C : BC;$$
$$\therefore W' : W :: A'C : AC.$$

Cor. 8. If a body is supported on a curve surface to which AC is a tangent at W, fig. 46, the effect will be the same as if it were supported on the plane AC. For the equilibrium depends only upon the direction of the surface at the point W, which is the same in the plane and in the curve.

Cor. 9. If the weight W, instead of being in contact with the plane in one point only, touch it in a finite portion, or the whole, of its length AC, (as in fig. 48) the proportion of the power and weight will be the same as before, supposing the friction not to be considered. For the weight supported at each point will be in the same proportion to the part of the power which supports it; and hence the whole weight will have this proportion to the whole power.

Cor. 10. Let the angle of inclination of the plane to the horizon (CAB) be α; the angle which the string makes with the plane (KWC) be ϵ: then

$$WK : KN :: \sin WNK : \sin KWN,$$
$$:: \sin CAB : \sin KWR,$$
$$:: \sin CAB : \cos KWC;$$
$$\text{or } P : W :: \sin \alpha : \cos \epsilon.$$

6. *The Wedge.*

109. A Wedge is a triangular prism; and, when applied as a mechanical power, is generally used to separate obstacles,

by introducing between them its edge and then thrusting it forwards. Thus, if ACc, fig. 49, be the end of a wedge, two objects EW, Ew, which have a tendency to rush together, may be separated by a force, (as a weight P,) applied at the back of the wedge, provided there be an immoveable obstacle at E. In the present Chapter we must consider the power as *in equilibrium* with the resistance, that is, P must be such a power as is just sufficient to prevent the wedge from being driven upwards, and not great enough to force it downwards.

In consequence of the immoveable obstacle at E, and of the nature of the object EW, the point W in the object, will, if it move at all, be compelled to move in a certain direction WU*. Whatever force tends to produce motion in W will be effective only so far as it acts in this direction. Thus, if WV be a force acting on the object W, it will be equivalent to WU, and to UV perpendicular to WU: of which the latter is counteracted by the immoveable obstacle at E, and WU is effective in opposing the resistance.

It is manifest, that the weight or resistance at W must be measured by the force which must be applied *immediately* at W to balance it. That is, if UW, uw, be the directions in which the points W, w will move if they move at all, and if we suppose W, w, to represent the forces which must be applied at those points in the directions WU, wu, to keep the parts EW, Ew in their present position when the wedge is removed; W, w, will also represent the resistances which are to be balanced by the wedge.

If we suppose the sides of the wedge to be perfectly smooth, their action at W, w, will necessarily be perpendicular to their surfaces, (Art. 34.)

This being premised, we can find the proportion of the power, and the weight or resistance. We shall take the case in which the wedge is isosceles, that is, when AW is equal to Aw, and the angles AWU, Awu, as also the resistances W, w, are equal. In this case the direction DA in which the power acts, must pass through the point A, and bisect the angle WAw.

* W will move in a curve to which WU is a tangent at W.

Prop. *In the wedge, to find the proportion of* P *and* W.

Draw OWV perpendicular to AW, and join Ow, which will be perpendicular to Aw, because the triangles OAW and OAw are equal. Join Ww meeting AO in M; therefore $WM = wM$.

Let WV, equal to WO, represent the action of the wedge perpendicular to its side; WV is equivalent to a force WU (which immediately opposes the resistance, and is therefore equal to it,) and a force UV perpendicular to WU, which is counteracted by the obstacle at E. Hence, UW representing the resistance, $WO = WV$ may represent the re-action on the side of the wedge. Similarly, the re-action on the wedge at w, arising from an equal resistance similarly applied, may be represented by wO. Also WO is equivalent to WM, MO; and wO, to wM, MO; of which WM, wM balance each other; and if the remaining forces MO, MO be balanced by a power $2P$ represented by $2MO$, the whole will be in equilibrium,

$$\therefore 2P : W :: 2MO : WU.$$

Cor. 1. If WU coincide with WV, or the resisting body be to be moved perpendicularly to AC, WMO, ADC will be similar triangles,

$$\therefore P : W :: MO : WV :: MO : WO :: DC : AC;$$

$$\therefore 2P : W :: Cc : AC.$$

Cor. 2. If WU be perpendicular to AD;

$$\therefore P : W :: MO : WU :: MO : MW :: DC : AD;$$

$$\therefore 2P : W :: Cc : AD.$$

Cor. 3. The action of the resistance upon the side AC is necessarily perpendicularly to AC*. The reason why W does not move in that direction is that it is also acted on by the resistance of an immoveable obstacle E.

* This is different from the way in which the wedge is sometimes considered, when the resistances are supposed to act in any direction, as for instance, parallel to AD. This is impossible; for if a body, as W, be pressed upon the side AC with a force parallel to AD, and with no lateral force, it will necessarily slide along AC, and the equilibrium cannot be established.

COR. 4. Let CAD, half the angle of the wedge, $=\alpha$, and UWV, the angle contained between WV perpendicular to AC, and WU the direction of the resistance, $=\iota$; W the resistance on each side; $2P$ the power,

$$2P : W :: \frac{2MO}{OW} : \frac{WU}{WV} \text{ because } OW = WV;$$

$$:: 2\sin OWM : \cos UWV,$$

$$:: 2\sin\alpha : \cos\iota,$$

$$\text{and } 2P : 2W :: \sin\alpha : \cos\iota,$$

where $2W$ is the whole resistance.

7. *The Screw.*

110. The general form of a screw is well known. It consists of a cylinder, as CD, fig. 50, on the surface of which is a projecting rib or *thread* which runs round the cylinder, and at the same time proceeds uniformly along the cylinder lengthways. This part of the instrument is inserted into a similar hollow cylinder AB which, with its thread, it exactly fits. In fig. 50, half of the internal and half of the external screw, are supposed to be removed, for the purpose of shewing its construction.

It is manifest that if the external screw be fixed, the internal one can only move by turning on its axis, by which means it will also move lengthways. If we suppose the vertical cylinder DC to be urged in the direction of its length by a weight W, it will be clear, by considering the form of the machine, that DC will descend, each point of the thread which is in contact with the external screw descending upon the inclined surface of the external thread, as upon an inclined plane. And the weight may be prevented from descending by a force P acting at an arm CM, which prevents the screw from turning round.

The form of the screw is such that when its axis is vertical, the inclination of the thread to the horizon is at every point the same. The thread may be considered as an inclined plane wrapped round the cylinder. Let, in fig. 51,

fhf' be a right-angled triangle, of which the base fof' is equal to the circumference of a horizontal section FoF of the cylinder. If then this triangle be wrapped round the cylinder so that fof' coincides with the circle FoF, the hypotenuse fnh will coincide with FnH, the thread of the screw. And FH will be parallel to the axis, and is called the *distance of two contiguous threads.*

PROP. *In a vertical screw, when a weight* W *is supported by a horizontal force* P *acting perpendicularly at the end of an arm* CM, P : W :: *dist. of two contiguous threads* : *circumference of the circle whose radius is* CM.

In fig. 51, let the internal screw, which sustains the weight W, be supposed to be supported by its thread resting on the fixed thread FGH of the external screw. Then we may suppose a portion of the weight to be supported at each portion of the thread, and the whole weight will be the sum of these portions. Let a weight w be supported at n, by means of the arm CM; let cnm be an arm equal to CM, and let a force p, acting horizontally, and perpendicularly to cm, support w. Then w will be sustained in the same manner as if it were upon the inclined plane fnh, for this plane and the thread FnH are in the same direction at the point n. And the effect of the force p is to produce a horizontal pressure on n, which prevents it from descending; let this force be q. Then we have, by the property of the lever,

$$p : q :: cn : cm;$$

$$:: \text{circumf. to rad. } cn : \text{circumf. to rad. } cm;$$

and, by the property of the inclined plane,

$$q : w :: f'h : ff' :: FH : \text{circumf. to rad. } cn;$$

$$\therefore p : w :: FH : \text{circumf. to rad. } cm :: D : C, \text{ suppose};$$

$$\therefore p = \frac{D}{C} w.$$

In the same manner, let the weight w' be supported at any

other point by p' acting at the end of an arm $= CM$; w'' by p'', &c. And we shall have

$$p' = \frac{D}{C}w', \quad p'' = \frac{D}{C}w'', \text{ \&c.}$$

$$\therefore \ (p + p' + p'' + \text{\&c.}) = \frac{D}{C}(w + w' + w'' + \text{\&c.}).$$

And the sum of all the partial weights will be the whole weight supported; and the power $p + p' + p'' +$ &c. acting at M will produce the effect of the separate powers p at cm, &c. (Art. 48.). Hence,

$$p + p' + p'' + \text{\&c.} = P, \quad w + w' + w'' + \text{\&c.} = W;$$

$$\text{and } P = \frac{D}{C}W, \text{ or}$$

$P : W :: D : C ::$ dist. of two threads ; circumf. to rad. CM.

Cor. 1. Instead of supposing the screw to support a weight W acting vertically, we may suppose it employed to produce a pressure W in any direction, and the proportion will be the same as before.

Cor. 2. In fig. 50, the form of the thread which is wraped round the cylinder is such that its section through the axis of the screw gives a rectangular profile, with sides parallel and perpendicular to the axis. But the mechanical advantage will be the same, whatever be the form or depth of this profile, so long as the inclination of the thread is the same.

Cor. 3. The proportion of the power and weight would be the same, if the internal screw were fixed, and the external one, carrying the weight, were moveable.

Cor. 4. The diameter of the cylinder does not effect the proportion of P to W, so long as the distance of the threads remains the same.

8. *Combination of Mechanical Powers.*

111. The *advantage* of a simple machine is the number expressing the multiple which the weight or effect produced is of the power or force producing it. The advantages of the different simple mechanical powers are as follows; (see the preceding Articles).

$$\text{Of the Lever, advantage} = \frac{\text{arm of the power}}{\text{arm of the weight}},$$

$$\text{Wheel and axle}\ldots\ldots\ldots\ldots \frac{\text{radius of wheel}}{\text{radius of axle}}.$$

$$\text{Toothed wheels}\ldots\ldots\ldots\ldots \frac{\text{n}^{\text{r}}\text{. of teeth of wheel}}{\text{n}^{\text{r}}\text{. of teeth of pinion}} \left\{\begin{array}{l}\text{nearly, when}\\ \text{the teeth are}\\ \text{small.}\end{array}\right\}$$

$$\begin{array}{ll}\text{Single moveable Pully}\ldots\ldots\ldots\ldots\ldots\ldots 2 & \\ \text{First System}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2^n & \\ \text{Second System}\ldots\ldots\ldots\ldots\ldots\ldots\ldots n & \\ \text{Third System}\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2^n-1 & \end{array} \left\{\begin{array}{l}\text{when the strings}\\ \text{are parallel and}\\ \text{the pullies with-}\\ \text{out weight.}\end{array}\right\}$$

$$\text{Inclined Plane}\ldots\ldots\ldots\ldots \frac{\text{length of plane}}{\text{height of plane}} \left\{\begin{array}{l}\text{when the power acts}\\ \text{parallel to the plane.}\end{array}\right\}$$

$$\text{Wedge}\ldots\ldots\ldots\ldots\ldots\ldots \frac{\text{side of wedge}}{\text{back of wedge}} \left\{\begin{array}{l}\text{when the resistance}\\ \text{acts perpendicularly}\\ \text{to the side.}\end{array}\right\}$$

$$\text{Screw}\ldots\ldots\ldots\ldots \frac{\text{circumf. desc}^{\text{d}}\text{. by power}}{\text{distance of threads}} \left\{\begin{array}{l}\text{when the power}\\ \text{acts in a plane}\\ \text{perpendicular to}\\ \text{the axis.}\end{array}\right\}$$

From these the *advantage* of compound machines may be found.

112. PROP. *The* advantage *of a combination is found by multiplying together the* advantages *of the separate machines.*

This may be shewn without difficulty in any particular case.

Fig. 52, represents a combination of the screw, the wheel and axle, the pully, and the inclined plane. A winch BC turns a cylinder CD, on which is the thread of a screw. This thread works in the teeth of a wheel ED, which has an axle EF. The cord which passes round this axis acts on a system of pullies of the second kind, attached to the fixed point G. This system draws a mass W up the inclined plane GH.

Let P be the power at B, acting perpendicularly to CB;

$$\frac{\text{pressure at } D}{P} = \frac{\text{circ. desc}^{\text{d}}\text{. by } B}{\text{dist. of threads}} = n \text{ suppose},$$

$$\frac{\text{pressure at } F}{\text{pressure at } D} = \frac{\text{rad. of wheel } ED}{\text{rad. of axle } EF} = n',$$

$$\frac{\text{force at } H}{\text{tension at } F} = \text{number of strings at } H = n'',$$

$$\frac{W}{\text{force in } GH} = \frac{\text{length of plane}}{\text{height of plane}} = n''';$$

$$\therefore \frac{W}{P} = nn'n''n'''.$$

For the sake of example, take the following numbers.

Let $CB = 18$ inches, distance of threads $= 1$ inch;

$\therefore$ circumf. by $B = 113$ inches nearly; $n = 113$.

$ED = 2$ feet, $EF = 6$ inches; $\therefore n' = 4$.

Number of strings at $H = 4$; $\therefore n'' = 4$.

Inclination of plane $= 30^0$; $\therefore n''' = 2$;

$$\therefore \frac{W}{P} = 113 \,.\, 4 \,.\, 4 \,.\, 2 = 3616.$$

Hence, on such a machine a force of 3 pounds would raise a weight of 10,000 pounds.

113. The following example of a combination of levers has some remarkable properties.

In fig. 59*a*, let CA, AB, BD be three bars moveable in the plane of the paper about centers at C and D, and about joints

at A and B. A force acts at E in the direction EF, and produces a pressure at B. Let this pressure be exerted in the direction CB, against a body placed between B and the immoveable obstacle G; and let it be required to determine the magnitude of the pressure. Draw CM perpendicular on EF; CN, DO on AB; DL on CB.

Let the force which acts in EF be P; and let W be the pressure produced at B in the direction CB. The lever CA communicates pressure to the lever DB by means of the bar AB; and the pressure thus communicated is in the direction of the length AB. Let Q be this pressure. The force Q acting on the lever CA in the direction BA balances the force P acting in EF: hence

$$\frac{P}{Q} = \frac{CN}{CM}.$$

Also the force Q acting in the direction AB on the lever DB produces the pressure W: hence

$$\frac{Q}{W} = \frac{DL}{DO};$$

$$\therefore \frac{P}{W} = \frac{CN \cdot DL}{CM \cdot DO}.$$

If we suppose CA, AB, to be in the same straight line, CN will vanish, and W will be infinitely greater than P. And if, at the moment when the pressure W is exerted, CA, AB be nearly a straight line, the pressure will be very great in comparison of the force employed, and may be increased without limit.

A combination depending upon principles nearly similar is used in the Stanhope and the Columbian printing presses, for the purpose of pressing together the types and the paper. The considerations by which its convenience is shewn belong partly to the following articles. It will there be seen that when W is very great compared with P, the velocity of B's motion must in the same proportion be small compared with P's. But by the contrivance above described, B's velocity is not small compared with P's, till CA, AB are nearly a straight line. Hence

B moves with a convenient rapidity while it is going toward the position in which the great pressure is to be exerted, and then only moves very slow, when it is come into this position and is actually exerting the pressure.

SECTION III.

GENERAL PROPERTY OF THE MECHANICAL POWERS.

114. By means of machines a given force may be made to overcome any resistance, or to raise any weight whatever: but it will be shewn in the following Propositions that what is gained in power is lost in time: that is, in proportion as the force which we exert to move a weight is increased by machinery, the velocity with which the weight moves is diminished.

When bodies move through spaces which have always the same proportion, their velocities have this proportion also. But when the proportion of the space is variable, we may suppose the bodies to describe very small spaces, and the ratio of these will be the ratio of the velocities *ultimately*, that is, by supposing the spaces to be diminished without limit.

Prop. *To find the velocity of a body estimated in a given direction.*

Let a point W, fig. 56, move in a direction Ww. Let WP, wp be parallel lines drawn in any other direction; and let wn be perpendicular on WP. If Ww represent the body's velocity in the direction of its motion, Wn will represent its velocity estimated in the direction WP.

Also we have $Wn = Ww \cdot \cos PWw$.

115. Prop. *In any of the mechanical powers, we shall have power : weight :: weight's velocity in the direction of its action : power's velocity in the direction of its action.*

We shall prove this by an enumeration of the cases of the different mechanical powers.

1. *The Lever.*

116. Let ACB, fig. 53, be a lever, acted on in directions AP, BW, by forces, P, W: and let CM, CN, be perpendiculars on the directions of the forces. Let the lever move through a small angle into the position aCb. A and B will describe circular arcs, Aa, Bb, which will be as the velocities of the points A and B, and being very small, may ultimately be taken for straight lines; and hence if am, bn be drawn perpendicular to AP, BW, Am, Bn will be as the velocities in the directions of the forces, by last Article.

Now, considering Aa as a straight line, CAa will ultimately be a right angle; hence,

$$CAM + aAm = \text{a right angle} = CAM + ACM,$$

and taking away CAM, $aAm = ACM$. Hence, the triangles CAM, Aam are similar. In the same way CBN and Bbn are similar. Also the angle aCb being equal to ACB, taking away aCB, we have $ACa = BCb$; and $CA = Ca$, $CB = Cb$, therefore the triangles ACa, BCb, are similar. Hence, we have these proportions,

$$Am : Aa :: CM : CA,$$

$$Aa : Bb :: CA : CB,$$

$$Bb : Bn :: CB : CN.$$

Hence, compounding the proportions,

$$Am : Bn :: CM : CN$$

$$:: W : P, \text{ by Art. 20.}$$

$\therefore$ P's velocity : W's velocity :: W : P.

2. *The Wheel and Axle.*

117. If the wheel and axle, fig. 38, turn through any angle, it is manifest that the arcs described by the points M and N are as CM and CN. But the arcs described are equal to the length of string wrapped at one point and

unwrapped at the other, and are therefore as the velocities of P and W. Hence

$$P\text{'s velocity} : W\text{'s velocity} :: CM : CN$$
$$:: W : P, \text{ by Art. 102.}$$

3. *Toothed Wheels.*

118. Let A, B, fig. 54, be wheels which turn each other in any manner by means of their circumferences. If they are toothed wheels, we suppose the teeth small, so that the point of contact may be conceived to be at O, in the line joining their centers. We will suppose also that the power and weight hang from equal axles CE, DF. In this case $P : W :: CO : DO$, by Art. 103.

Now, let the wheels turn through a small angle, so that the points which were in contact at O, come to m and n. Om and On will be equal, because they have been applied to each other. And drawing meC meeting the circle CE in e, and nDf meeting the circle DF in f, Ee and Ff will be the spaces ascended and descended by P and W. And we have, by the similar sectors in the figure,

$$Ee : Om :: CE : CO,$$
$$On\,(= Om) : Ff :: DO : DF\,(= CE);$$
$$\therefore Ee : Ff :: DO : CO,$$
$$\text{or } P\text{'s velocity} : W\text{'s velocity} :: W : P.$$

Cor. Ee, Ff are as the angular velocities of the wheels A and B. Hence, in wheels which work in each other, the angular velocities are inversely as the radii. Hence also the number of revolutions in a given time will be inversely as the radii.

4. *Pullies.*

119. (1.) *In the single moveable pully with parallel strings*, if the weight W, fig. 40, be raised through any space, as 1 inch, each of the strings, AP, BC, will be shortened one

inch at the lower end, and hence the power P will move upwards through 2 inches. Hence,

$$P\text{'s velocity} : W\text{'s velocity} :: 2 : 1 :: W : P.$$

120. (2.) *In the single moveable pully with strings not parallel;* fig. 55, let the pully at A be considered as a point. Let CAK be the position of the string, and let it be moved into the position CaK, so that W ascends through the small space Aa, and P descends through Pp. Take Km, Cn equal to Ka, Ca respectively; and $Am + An$ is the quantity by which the string CAK is shortened, and therefore the quantity by which KP is lengthened, or $Pp = Am + An$. Now when the angle AKa is very small, am may be considered as ultimately perpendicular on AK, and an on AC: hence

$$Am = Aa \cos aAm = Aa \cos \alpha, \text{ if } \alpha = KAa.$$

$$\text{Similarly, } An = Aa \cos \alpha;$$

$$\therefore Pp = 2Aa \cos \alpha;$$

$$\therefore Pp : Aa :: 2 \cos \alpha : 1;$$

$$\text{or } P\text{'s velocity} : W\text{'s velocity} :: W : P, \text{ by Art. 104.}$$

If the pully be of finite magnitude, as in fig. 41, since, when the change of position is small, the strings KA, CB, may be considered as remaining parallel to themselves, the part of the string AB which is wrapped round the pully is not altered; and hence the length of the space described by P is not altered on this account.

121. (3.) *In the first system of pullies*, fig. 42, if the weight W be raised through any space, as 1 inch, the pully A_2 is, as in the single moveable pully, raised 2 inches; hence, for the same reason, the pully A_3 is raised 2.2 inches; and similarly, a succeeding pully would be raised 2.2.2 inches; and so on to P, which will, by this reasoning be lowered 2^n inches: hence

$$P\text{'s velocity} : W\text{'s velocity} :: 2^n : 1 :: W : P.$$

122. (4.) *In the second system of pullies*, fig. 43, if the weight W be raised 1 inch, each of the strings by which

the lower block hangs will be shortened 1 inch; and hence the whole length of the string between the blocks will be shortened n inches, and P will descend n inches; hence

$$P\text{'s velocity} : W\text{'s velocity} :: n : 1 :: W : P.$$

COR. In this system, while 1 inch passes round the pully A_1, 2 inches pass round the pully B_1, 3 round A_2, 4 round B_2, &c.

Hence, if the radii of A_1, B_1, A_2, B_2, &c. be as 1, 2, 3, 4, the velocities of their circumferences will be as the radii, and therefore the angular velocities will be equal; and hence A_1, A_2 may be on the same axis, and may form one mass, and similarly B_1 and B_2 may be united on one axis, as in fig. 44.

123. (5.) *In the third system of pullies*, fig. 45, let the weight be raised 1 inch; then the pully A_2 will descend 1 inch: on this account the pully A_1 will descend 2 inches; and also on account of C_2 being raised 1 inch, A_1 will descend 1 inch; therefore it will descend $2+1$ inches. Again, on this account P will descend $2(2+1)$ or 2^2+2 inches, and 1 inch more in consequence of C_1 being raised 1 inch; hence, P will descend 2^2+2+1 inches $=2^3-1$ inches; hence,

$$P\text{'s velocity} : W\text{'s velocity} :: 2^3-1 : 1 :: W : P;$$

and similarly for any number of pullies.

5. *The Inclined Plane.*

124. Let W, fig. 56, be raised through a small space Ww, WP being supposed parallel to wp. Draw WE horizontal, and wm, wn perpendicular to WE, WP. Therefore Wn, wm are ultimately as the velocities in the directions of the power and weight. But if $CAB = wWm = a$, and $CWP = \epsilon$, we have

$$Wn : wm :: Ww \cos\epsilon : Ww \sin a$$

$$:: \cos\epsilon : \sin a;$$

or P's velocity : W's velocity :: W : P, Art. 108. Cor. 10.

6. *The Wedge.*

125. Let an isosceles wedge ADC, fig. 57, in which AD is the line bisecting the back, move in the direction of the line DA through a small space Aa. Let the point W move through a space Wn, in the direction WU, making an angle ι with WV, which is perpendicular to the side AC. Then we shall have

$$Wn = \frac{Wm}{\cos\iota} = \frac{Aa \sin a}{\cos\iota}, \quad a \text{ being} = DAC;$$

$$\therefore Aa \text{ or } Dd : Wn :: \cos\iota : \sin a,$$

or P's velocity : W's velocity :: $W : P$, by Art. 109. Cor.

7. *The Screw.*

126. If M, fig. 51, make a whole revolution with a uniform velocity, W will rise with a uniform velocity through the distance of two contiguous threads; and the space described by P, estimated in a horizontal direction (in which direction the force is supposed to act) is the circle whose radius is CM; hence

P's velocity : W's velocity :: circle rad. $= DE$: distance of threads :: $W : P$.

8. *Any Combination of Machines.*

127. In any combination of these machines, the ratio of the power's velocity to the weight's velocity will be found by multiplying the ratios which obtain in the machines of which it is composed; and the ratio of the weight to the power is found by multiplying the ratios in each of the component machines, which ratios have been shewn to be the same as the former; hence the resulting ratios will be the same; and hence in all combinations of machines by which a power P sustains a weight W, if the machine be put in motion through a very small space,

P's velocity in its direction : W's velocity in its direction :: $W : P$.

COR. 1. Hence we have $P.P$'s velocity $= W.\ W$'s velocity.

A weight multiplied into its velocity is called its *Momentum:* hence P's momentum $= W$'s momentum.

COR. 2. If $P.\ P$'s velocity $= W.\ W$'s velocity, P and W will balance: for if not, let P and W' balance on the same machine: then $P.\ P$'s velocity $= W'.\ W'$'s velocity: and the velocity of W' is the same as that of W, so long as the machine remains the same. Hence $W' = W$, and therefore P and W balance.

CHAPTER IX.

EXAMPLES OF THE CENTER OF GRAVITY FROM THE FORMER EDITIONS.

128. EXAMPLES *of finding the center of gravity.*

EX. 1. To find the center of gravity of *a straight line;* supposed to be of uniform thickness and density.

A straight line will balance itself about its *middle point* in every position: this point is therefore the center of gravity.

EX. 2. To find the center of gravity of *a parallelogram,* as $ABCD$, fig. 63.

Bisect the opposite sides AB and DC in E and F, and the opposite sides AD and BC in H and K; and let the lines EF, HK meet in G: G is the center of gravity.

For the parallelogram may be conceived to be made up of lines parallel to AB, as for instance PMQ; and since $PM = AE = EB = MQ$, each of these lines, as PQ, will balance in every position on the point M, that is, on the line EF: hence the whole parallelogram will balance on the line EF. Similarly the whole parallelogram will balance on the line HK. Hence it will balance in every position on the point G; which is therefore the center of gravity.

EX. 3. To find the center of gravity of *a triangle;* as ABC, fig. 64.

Bisect AB in E, and AC in F; join CE, BF; the intersection G is the center of gravity.

For the triangle may be conceived to be made up of lines parallel to AB, as PQ: and we have by similar triangles,

$$\frac{PM}{AE} = \frac{MC}{EC} = \frac{MQ}{EB}, \text{ and since } AE = EB,\ PM = MQ.$$

Hence each of the lines PQ will balance on the line CE in every position, and therefore the whole triangle will balance on that line. Similarly the whole triangle will balance on BF in every position; and hence it will balance in every position on the intersection G, which is therefore the center of gravity.

Join FE; and since $AE = \frac{1}{2}AB$, and $AF = \frac{1}{2}AC$, EF is parallel to BC; hence by similar triangles, AEF, ABC,

$$\frac{EF}{BC} = \frac{AE}{AB} = \frac{1}{2},$$

whence by similar triangles EFG, CBG,

$$\frac{EG}{GC} = \frac{EF}{BC} = \frac{1}{2}; \therefore GC = 2EG; \therefore EC = 3EG,$$

$$\text{hence } EG = \frac{1}{3}EC, \text{ and } GC = \frac{2}{3}EC.$$

COR. 1. If we call the sides opposite to A, B, C, a, b, c respectively, and CE, e; since $AE = EB = \frac{c}{2}$,

$$e^2 = \frac{2a^2 + 2b^2 - c^2}{4}\text{*}.$$

$$\text{Hence } CG = \frac{2}{3}e = \frac{\{2(a^2 + b^2) - c^2\}^{\frac{1}{2}}}{3}.$$

* In any triangle ABC, Fig. 64, if a side AB be bisected in E; retaining the letters in the text, we have

$$\text{in triangle } ACE,\ b^2 = \left(\frac{c}{2}\right)^2 + e^2 - 2.\frac{c}{2}.e\cos CEA,$$

$$\text{in triangle } BCE,\ a^2 = \left(\frac{c}{2}\right)^2 + e^2 + 2.\frac{c}{2}.e\cos CEA,$$

add, and we have

$$a^2 + b^2 = \frac{c^2}{2} + 2e^2;$$

whence the formula in the text.

COR. 2. If we call GA, GB, GC, h, k, l, respectively, we shall have $l = \frac{2}{3} e$; whence

$$3l = \{2(a^2 + b^2) - c^2\}^{\frac{1}{2}}.$$

Hence $9l^2 = 2a^2 + 2b^2 - c^2$;

similarly, $9h^2 = 2b^2 + 2c^2 - a^2$;

$9k^2 = 2c^2 + 2a^2 - b^2$; and, by addition,

$$9(h^2 + k^2 + l^2) = 3(a^2 + b^2 + c^2);$$

or $3(h^2 + k^2 + l^2) = a^2 + b^2 + c^2$.

COR. 3. If three equal bodies be placed in the angles of a triangle, the center of gravity of these bodies is the same as the center of gravity of the triangle.

COR. 4. To find the center of gravity of any *polygon*, divide it into triangles; and supposing each of these collected at its center of gravity, find the center of gravity of the whole; which, by Art. 86, will be the center of gravity of the polygon.

EX. 4. To find the center of gravity of *a quadrilateral* $ACBC'$, fig. 65, which has two adjacent sides equal, and also the two other adjacent sides equal: $AC = BC$, and $AC' = BC'$.

Join CC', which will bisect AB in D, and will be perpendicular to AB. Let E be the center of gravity of ABC and F of ABC; if we take G so that

$$EG : FG :: ABC' : ABC :: DC' : DC,$$

G will be the center of gravity:

and hence $EG : EF :: DC' : CC'$, and $EG = \frac{DC' . EF}{CC'}$.

Let $DC = c$, $DC' = c'$; $\therefore DE = \frac{c}{3}$, $DF = \frac{c'}{3}$; $\therefore EF = \frac{c + c'}{3}$;

$$\therefore EG = \frac{c'}{c + c'} \frac{c + c'}{3} = \frac{c'}{3}; \therefore DG = DE - EG = \frac{c}{3} - \frac{c'}{3} = \frac{c - c'}{3}.$$

Cor. Similarly, if C and C' were both on the same side of AB, we should have

$$DG = \frac{c + c'}{3}.$$

Ex. 5. To find the center of gravity of *a quadrilateral* $ABDC$, fig. 66, of which two sides AB, CD are parallel.

Bisect AB, CD in H and K, and join HK; all lines parallel to AB will balance on HK, and therefore the center will be in that line. Join BC, CH, BK; and take $CE = \frac{2}{3} CH$, and $BF = \frac{2}{3} BK$; E and F will be the centers of gravity of the triangles ABC, DBC, which may, by Cor. to Art. 86, be considered as collected at those points. Hence, if EM, FN be parallel to HK, and G be the center of gravity, it may easily be shown that

$$GH = \frac{\text{triangle } ABC . EM + \text{triangle } BCD . FN}{\text{triangle } ABC + \text{triangle } BCD}.$$

Let CL be parallel to KH, CI perpendicular to AB; therefore, by similar triangles,

$$\frac{EM}{CL} = \frac{HE}{HC} = \frac{1}{3}; \quad \frac{FN}{KH} = \frac{BF}{BK} = \frac{2}{3};$$

$$\therefore EM = \tfrac{1}{3} CL = \tfrac{1}{3} KH, \; FN = \tfrac{2}{3} KH;$$

$$\therefore GH = \frac{\frac{1}{2} AB . CI . \frac{1}{3} KH + \frac{1}{2} CD . CI . \frac{2}{3} KH}{\frac{1}{2} AB . CI + \frac{1}{2} CD . CI}$$

$$= \frac{AB . KH + 2CD . KH}{3(AB + CD)}.$$

If $AB = a$, $CD = b$, $KH = c$,

$$GH = \frac{c}{3} \, \frac{a + 2b}{a + b}.$$

Cor. When $b = 0$, this gives $GH = \frac{c}{3}$, and the trapezium becomes a triangle.

Ex. 6. To find the center of gravity of *a pyramid whose base is a triangle ABC*, fig. 67, and whose vertex is O.

Bisect BC in D, join AD, OD, and take $DE = \frac{1}{3}DA$, $DF = \frac{1}{3}DO$; join OE, AF; their intersection G will be the center of gravity of the pyramid.

The pyramid may be conceived to be made up of planes parallel to ABC, as PQR; E is the center of gravity of the triangle ABC, and N, where OE meets PQR, will be the center of gravity of PQR; as may easily be shewn. Hence each of the triangles PQR will balance on the line OE, and hence the whole pyramid will balance in any position about OE. Similarly, the whole pyramid will balance on the line AF: hence it will balance in every position on the intersection G, which is therefore the center of gravity.

By similar triangles,

$$\frac{EF}{AO} = \frac{ED}{AD} = \frac{1}{3}; \text{ and } \frac{EG}{GO} = \frac{EF}{AO} = \frac{1}{3};$$

$$\text{hence } GO = 3EG; \quad EG = \frac{1}{4}EO, \text{ and } GO = \frac{3}{4}EO.$$

Cor. 1. Bisect AO in H, and draw HK parallel to OE; hence by similar triangles,

$$\text{since } AH = \frac{1}{2}AO, \therefore AK = \frac{1}{2}AE = DE; \therefore DK = 2DE.$$

$$\text{Also } HK = \frac{1}{2}OE, \text{ and } GE = \frac{1}{4}OE; \therefore HK = 2GE.$$

Hence $DE : DK :: GE : HK$, and DGH is a straight line bisected in G.

Hence we have this theorem: if in a triangular pyramid we bisect two edges which do not meet, and join the points of bisection, and bisect the joining line; the last bisection is the center of gravity of the pyramid.

COR. 2. To find OG, let the edges of the pyramid adjacent to O, viz. OA, OB, OC be a, b, c; and the others BC, CA, AB, a', b', c', respectively: also let AD, OD, OE, be e, f, g.

Then we shall have*

$$g^2 = \frac{6f^2 + 3a^2 - 2e^2}{9}.$$

And by Cor. 1, to Ex. 3, we have

$$e^2 = \frac{2b'^2 + 2c'^2 - a'^2}{4},$$

$$f^2 = \frac{2b^2 + 2c^2 - a'^2}{4}.$$

Hence, by substitution,

$$g^2 = \frac{3(a^2 + b^2 + c^2) - (a'^2 + b'^2 + c'^2)}{9}.$$

And $OG = \frac{3}{4}\cdot g = \frac{1}{4}\left\{3(a^2 + b^2 + c^2) - (a'^2 + b'^2 + c'^2)\right\}^{\frac{1}{2}}$.

COR. 3. If we join the center of gravity with each of the four angles O, A, B, C, and call the distances h, k, l, m, respectively, we shall have h^2, k^2, l^2, m^2, by formulæ easily derived from the preceding; and adding these together, we shall have

$$4(h^2 + k^2 + l^2 + m^2) = a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2.$$

* In any triangle AOD, Fig. 67, if a side AD be divided so that DE is $\frac{1}{3}$ of DA; retaining the letters in the text, we have

$$\text{in triangle } DOE,\ f^2 = \left(\frac{e}{3}\right)^2 + g^2 + 2\cdot\frac{e}{3}\cdot g\cos OEA;$$

$$\text{in triangle } AOE,\ a^2 = \left(\frac{2e}{3}\right)^2 + g^2 - 2\cdot\frac{2e}{3}\cdot g\cos OEA.$$

Add twice the first to the second, and we have

$$2f^2 + a^2 = \frac{6e^2}{9} + 3g^2;$$

whence the formula in the text.

Cor. 4. Since EG is $\frac{1}{4}$ of EO, it is manifest that if we draw parallel lines through G and O, meeting the base, the distance of G from this plane will be $\frac{1}{4}$ of the distance of O.

Ex. 7. To find the center of gravity of *any pyramid*, whose base is a polygon $ABCDE$, fig. 68, and vertex O.

The polygon may be divided into triangles by lines drawn from one angle to another; and if planes pass through these lines and through the vertex, the pyramid will be divided into triangular pyramids. If a plane be drawn parallel to the base, at a distance equal to $\frac{1}{4}$ of the altitude of the pyramid, by Cor. 4 to last example, the center of gravity of each of the triangular pyramids, and therefore of the whole pyramid, will be in this plane. But if we join O with F the center of gravity of $ABCDE$, it will appear, as in the last example, that the center of gravity will be in this line. Hence it will be in the point G where the line meets the plane. Also it is manifest that

$$FG = \tfrac{1}{4} FO, \quad \text{and} \quad OG = \tfrac{3}{4} OF.$$

Cor. If the number of sides of the polygonal base of the pyramid be increased without limit, the method of finding the center of gravity remains the same. Hence it will be true in the case to which we thus approximate, that is, that of a *conical body with a curvilinear base.* In all such cases we must find the center of gravity by measuring from the vertex $\frac{3}{4}$ of the line which joins that point with the center of gravity of the base.

Ex. 8. To find the center of gravity of *a frustum of a pyramid;* cut off by a plane parallel to the base.

The two ends will be similar figures; let a, b, be homologous sides of the larger and smaller end. Also let the centers of gravity of the two ends be joined, and let the line which joins them be called the axis and $b = c$. Then the

center of gravity will be in the axis, and it may be shewn, as in Ex. 5, that its distance from the larger end along this side will be

$$\frac{c}{4} \cdot \frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

COR. The same will be true of the *frustum of a cone;* a, b, representing the radii, or any homologous lines, in the two ends.

Ex. 9. *In any machine kept in equilibrium by the action of two weights, if an indefinitely small motion be given to it, the center of gravity of the weights will neither ascend nor descend.*

It is easy to shew this independently, in each of the mechanical powers.

In the straight lever, the centre of gravity is at the fulcrum, and remains fixed however the lever be moved.

In the wheel and axle, fig. 38, the centre of gravity of P and W is at G, in the vertical line passing through the center C, and if P descends, W ascends, and G remains fixed, as if PGW were a lever.

In the toothed wheels, fig. 54, if P ascends W descends; and the center of gravity G remains fixed in a point G, such that

$$PG : WG :: DO : CO.$$

In the systems of pullies, fig. 41, 42, 43, 44, 45, if we join P and W, and take $PG : WG :: W : P$, G will be the center of gravity; and if P descend W will ascend, so that P's descent : W's ascent :: $W : P :: PG : WG$; whence G remains fixed.

In the inclined plane, fig. 58, when the force is parallel to the plane, let P support W: and let P, W, be their situations when they are in the same horizontal line. Let P descend to p, and W ascend to w; $\therefore$ $Pp = Ww$: join wp meeting

WP in g; draw wm perpendicular on WP; now by similar triangles,

$$wg : pg :: wm : Pp :: wm : Ww : BC : AC :: P : W,$$

therefore g is the center of gravity of p, w. Hence the center has moved in the horizontal line Gg; and this is true whatever be the space described.

The wedge and screw do not generally act by gravity; when they do, the same property is easily proved.

DYNAMICS.

CHAPTER I.

ACCELERATING FORCE.

SECTION I.

THE FIRST LAW OF MOTION.

129. *In Dynamics we adopt the Ideas, Definitions, Axioms and Propositions of Statics, and establish others.*

Dynamics is the science of Force, considered as producing or altering motion. If the forces which act upon any body or bodies be not such as produce equilibrium, some motion or other must be the result. Dynamics is the science which has for its object to determine what the circumstances and properties of this motion will be.

Thus in Dynamics we consider Forces of the same kind as those which we considered in Statics; namely, *Pressures*. In Statics pressure produces equilibrium: in Dynamics pressure produces motion.

For example: a man puts a foot-lathe in motion by the *pressure* of his foot. A boy turns a wheel by means of a winch, *thrusting* and *pulling* it with his hands, that is, exerting forces of the nature of pressure. A horse draws a carriage, *pulling* by the traces; the *tension* of the traces is a pressure on the carriage. A stream of water *presses* the floats of a mill-wheel, and thus puts it and keeps it in motion. The gas generated by the explosion of gunpowder *presses* forwards the ball in the barrel of a gun and communicates velocity to it. The steam generated in the boiler of a steam-engine, being admitted into the cylinder, *presses* the piston and moves it. The piston moves or keeps in motion the fly-wheel, by *pushing* and *pulling* it alternately, that is, by pressure. The fly-wheel *presses* the rest of the machinery when it is disposed to slacken its speed, and thus equalizes the velocity.

130. Axiom. *Pressure among bodies in motion implies an action, and an equal and opposite re-action.*

The universal equality of action and re-action in all the mechanical conditions and changes of matter holds in the case of motion, for the same reasons as in the case of rest. The general axiom, that re-action is equal and opposite to action, is necessarily true; and the interpretation of action and of re-action of which we now speak, namely pressure, is the same in Statics and in Dynamics, as we have just said.

For example: the treadle of the foot-lathe presses the foot up any moment, just as much as the foot presses the treadle down. The traces of a carriage pull the horse back with the same tension with which the horse pulls the carriage forwards. In addition to this, the horse pushes the ground backwards and thus, by the re-action, pushes himself forwards. The pressure by which the fall of the water on a mill-wheel accelerates the wheel, is accompanied by an opposite pressure which retards the water. The gas which gives a velocity to the ball in a gun, exerts a re-action on the gun itself, and thus produces the recoil. The steam in the cylinder of a steam-engine, which presses the piston, presses with equal force the other end of the cylinder. The pressure which the connecting rod exerts on the fly-wheel produces a re-action upon the piston and the cylinder, which re-action makes it necessary that the cylinder should be very firmly fixt.

131. All motion is performed in *Time:* and the time is measured by the number of *units* of time which it contains. The passage of time is marked by the events which take place in it; and those intervals in which there is no discoverable reason why they should be unequal, are supposed equal. The intervals thus taken as a standard are, in all countries, the natural day and its divisions. The unit of time may be any portion we choose: in theoretical Mechanics a *second* is generally taken for the unit.

132. *Velocity* is the measure of the degree in which a body moves quickly or slowly: that is, one body is said to have a greater velocity than another when it moves over a greater space in the same time, or an equal space in a less time.

When a body moves over equal spaces in equal successive times the motion is said to be *uniform;* and the velocity is *constant:* and is *measured* by the space described in a unit of time, as for instance, in one second.

In variable motions, it will be seen hereafter that the velocity is measured by the space which *would be* described in a unit of time, if the velocity were uniform.

133. PROP. *In uniform motions, the space described in any time is equal to the product of the numbers which express the velocity and the time.*

Let v be the velocity expressed in feet; then, by the last Article, v is the number of feet described in one second. And since the motion is uniform, $2v$ is the space described in two seconds; $3v$ in three seconds; and generally, tv in t seconds. If s be the space, $s = tv$.

If we suppose the space described in equal fractions of a second to be equal, this equation will also be true when t is a fractional or mixed number.

COR. Since $s = tv$, $v = \dfrac{s}{t}$.

Hence in uniform motions the quotient of the space by the time is constant, and measures the velocity.

Thus if a ship, sailing uniformly, move 10 miles in 1 hour, the velocity, measured by the space described in a second, is

$$\frac{10 \times 5280}{60 \times 60} = 14\tfrac{2}{3} \text{ feet.}$$

134. When the velocity is not constant, it can no longer be measured by the quotient of the space divided by the time; for these quotients will be different for different times. Thus if we suppose a body to fall by the force of gravity, this force being diminished by machinery so that the body shall fall from rest 16 feet in the first 4 seconds, it will move, not with a constant, but with an increasing velocity. And if we then measure the space described by this body in the 4 seconds succeeding the first four, we shall find it to be 48 feet; in three seconds from the end of the first four, the space would be 33 feet; in two seconds, 20; in one second, 9; in the half second immediately following the fourth, the space will be $4\frac{1}{4}$ feet, and in the quarter-second after the fourth, it will be $2\frac{1}{16}$. Hence we shall have the following values of the quotient of the space by the time, measuring from the beginning of the fifth second.

Values of t,	4″	3″	2″	1″	$\frac{1}{2}$″	$\frac{1}{4}$″,
of s,	48	33	20	9	$4\frac{1}{4}$	$2\frac{1}{16}$,
of the quotient $\frac{s}{t}$,	12	11	10	9	$8\frac{1}{2}$	$8\frac{1}{4}$.

The quotients, commencing at the beginning of the fifth second, go on increasing, and are larger as we take the time larger. And this must always be the case with an increasing velocity; for the space described, beginning from any time, will depend both upon the velocity at that time, and upon the augmentation of velocity which takes place afterwards.

Also the portion of the space which is due to this augmentation is smaller as the time of the motion is smaller. And if we approach nearer and nearer to the initial point of time, we approach nearer and nearer also to the velocity at that point of time. Hence the following definition.

135. Def. *The velocity at any point is measured by the limit of the quotient of the increment of the space, divided by the time beginning from that point.*

(The *limit* is to be taken by supposing the space and the time indefinitely diminished.)

Thus, in the above instance, if we were to suppose more minute values of t to be taken, as $\frac{1}{8}$, $\frac{1}{16}$, it would appear that the value of $\frac{s}{t}$ would always be greater than 8. But the excess above 8 might be diminished, by diminishing s and t sufficiently, so as to be made smaller than any assigned quantity. Hence 8 is the limit of the fraction $\frac{s}{t}$, and 8 feet measures the velocity of the body at the beginning of the 5th second.

Instead of taking the time immediately after the point considered, we may take the time immediately before it, and we shall have the similar results.

136. Prop. *In any motion, the velocity is measured by the space which would have been described in a unit of time, if the velocity had continued constant.*

Let the velocity be increasing, and let s' be the space from the given point, which would be described in the time t, if the velocity were to continue constant from that time; let $s' + s''$ be the space which is actually described in t. Then, by last Article, the limit of $\frac{s' + s''}{t}$ is the measure of the velocity. In this expression, s'' is the part which arises from the augmentation of the velocity after the body leaves the given point, and its effect diminishes perpetually as t diminishes. Hence in taking the limit, when t vanishes, the effect of s'' cannot appear. Therefore the limit of $\frac{s' + s''}{t}$ is the same as the limit of $\frac{s'}{t}$; and $\frac{s'}{t}$ measures the velocity.

When $t = 1$, s' is the space described in a unit of time, supposing the velocity to become constant. Hence the space so described in a unit of time measures the velocity.

And similarly, for a decreasing velocity.

137. When a material plane revolves about any fixt point in the plane (that is, revolves about an axis perpendicular to the plane), a line drawn from any material point in the plane to the fixt point, moves through various angles, and thus has an *angular velocity*. Angular velocity is constant or variable, and may be measured in the same manner as linear velocity.

DEF. *Angular velocity, when constant, is measured by the angle described in a unit of time; and when variable, is measured by the limit of the quotient of the increment of the space divided by the increment of the time.*

138. The motions of bodies in machines follow various rules, as to their velocity and direction, according to the construction of the machine. Some motions are *alternate* or oscillatory, certain pieces going backwards and forwards; as saws, planes, files, pistons. Others are continually in the same direction: these are generally rotatory motions, as those of the wheels in many machines. Again, in the latter case the velocity is often constant, or nearly so. In alternate motions the velocity is

generally greatest towards the middle of each *trip*, or passage backwards and forwards; and diminishes till it vanishes at the instant, when the *inversion* of the direction of motion takes place.

If we suppose a circular wheel to revolve uniformly in a vertical plane about a fixt center, a given point in its circumference will go up and down with such an alternate motion as has been described. The velocity at any moment is proportional to that horizontal chord of the circle which passes through the given point, as may be proved. The velocity is greatest when this chord is the horizontal diameter of the circle, and diminishes gradually when the point passes above or below this diameter.

Such a law of velocity, in which, a circle being described upon the whole space moved through as a diameter, the velocity at any point is as the corresponding chord perpendicular to the diameter, is sometimes called *the law of a cycloidal pendulum*, for a reason which will appear hereafter.

The velocity in this case is as the sine of the arc from the extreme point; for this sine is half the chord.

139. The consideration of the laws of velocity and direction of parts of machines in general, and of the modification of such laws in the communication of the velocity from one part to another, belongs to a separate science, which has been termed *Kinematics*, or *Mechanism*. The subject of the present treatise is, motions considered with reference to the forces which produce or alter them.

140. The direction and velocity of any motion are not changed, except by the action of extraneous force. This is one of the fundamental laws of motion; and is thus stated.

A body in motion, not acted upon by any force, will go on for ever with a constant velocity.

This law, according to the usual course of developement of our mechanical conceptions and convictions, must be established, and according to the actual historical progress of the

subject was established, by induction from experimental and observed facts.

The facts which are included in this induction are such as the following:—

(1) All motions which we produce, as the motions of a body thrown along the ground, and of a wheel revolving freely, go on for a certain time and then stop.

(2) Bodies falling downwards go on moving quicker and quicker as they fall farther.

It was attempted to explain these facts, by saying that motions such as (1) are *forced* motions, and motions such as (2) are *natural* motions; and that forced motions decay and cease by their nature, while natural motions, by their nature, increase and become stronger.

But this explanation was found to be untenable; for it was seen—(3) that forced motions decayed less and less by diminishing the obvious obstacles. Thus a body thrown along the ground goes farther as we diminish the roughness of the surface; it goes farther and farther as the ground is smoother, and farther still on a sheet of ice. The wheel revolves longer as we diminish the roughness of the axis; and longer still, if we diminish the resistance of the air, by putting the wheel in an exhausted receiver.

Thus a decay of the motion in these cases (1) is constantly produced by the obstacles. Also an increase of the motion in the cases (2) is constantly produced by the weight of the body.

Therefore there is in these facts nothing to shew that any motion decays or increases by its nature, independent of the action of external causes.

(3) By more exact experiments, and by further diminishing the obstacles, the decay of motion was found to be less and less; and there was in no case any remaining decay of motion which was not capable of being ascribed to the remaining obstacles.

Hence the facts are explained by introducing the *Idea* of Force, as *that which causes change in the motion* of a body; and the *Principle*, that *when a body is not acted upon by any force, it will move with a uniform velocity.*

141. When a body (considered as a point) moves freely (not being retained by any axis or any other restraint), and is not acted upon by any force, it will move in *a straight line.*

For since it is not acted on by any force, there is nothing to cause it to deviate from the straight line on any one side.

This principle, along with the one above stated, is included in the *First Law of motion*, which is, accordingly, this:

A body in motion not acted upon by any force, will go on in a straight line with a constant velocity.

This Law was further confirmed and illustrated by attending to the circumstances which showed that force was requisite to alter the direction or velocity of any motion: and by considering the consequences which follow, when we act without attending to this rule. Thus if a road make a sudden turn, a carriage passing too rapidly along it will be overturned, towards the convex side of the turn, showing the want of a force to change the direction of the motion. If a carriage moving rapidly stop too suddenly, a person sitting on the box will be thrown forwards, showing the want of a force to reduce him from motion to rest.

SECTION II.

UNIFORMLY ACCELERATED MOTION.

142. The first law of motion being proved, it follows, that if a body, considered as a point, move either in a curve line or in a straight line with a velocity not constant, it is acted upon by some external force: and the deviation from rectilinear and uniform motion depends upon the direction and magnitude of the force which acts upon the body.

The *direction* of a force is the straight line in which the force would cause a body to move, if it acted on the body at rest. When a force acts on a body already in motion, the motion which the force would communicate to the body at rest will be combined with the other motion which the body has, according to laws which will be mentioned hereafter. If a force act upon a body in motion, so that the direction of the force coincides with the direction of the body's motion, the body manifestly will not be made to deviate on one side or the other of the direction, but will go on in this straight line

with unaltered velocity. If a force act so as to make an angle with the direction of the motion of the body, it will cause the body to describe a curvilinear path, the concavity of the path being on the side towards which the force tends.

143. Since causes are measured by their effects, the *magnitude* of forces is measured by their effects; and the effect of forces which we consider in Dynamics is motion; hence the effect which we have to take as the measure of force is velocity, with or without regard to the quantity of matter moved. The abstract quantity which we call *force*, may be defined in various ways for various purposes; and, in fact, we shall take different definitions of force in this and the succeeding chapters; as the definition of Accelerating Force, of Moving Force, of Vis Viva or Living Force, of Labouring Force. In the present chapter we leave out of our account the quantity of matter moved, and consider velocity alone. Hence forces are greater or less as they produce a greater or less velocity in the same time.

144. DEF. *Accelerating force is force measured by the velocity which, in a given time, it would produce in a body.*

When a body is acted upon by a continuous force, as pressure or attraction, the velocity communicated to the body goes on increasing as the force acts for a longer time. Thus, if a stone fall from rest during one second, and another stone fall during two seconds, the velocity of the latter stone, upon which gravity has acted for a longer time, will be the greater of the two. Similarly, if we produce velocity by the continued action of the hand, as when we turn by hand a machine carrying a fly-wheel; or by means of a spring, as when a bow impels an arrow; the velocity goes on increasing so long as the operation of the force continues. Now we may, at any point of time, suppose the action of force to cease; and, by the first law of motion, the body would then go on with the velocity already acquired: and if, after this, we suppose the force again to begin to act in the direction of the motion, an additional velocity will be communicated. Thus force produces a velocity in a body at rest, and adds velocity to the motion of a body already moving; and if the force be sup-

posed to act for any time, it is adding velocity during the whole of that time; and the velocity produced at last is the aggregate of all the successive additions.

If when a body is thus continually acted upon by an accelerating force in the direction of its motion, the velocity added be equal in equal times, the motion is said to be *uniformly accelerated;* the force is said to be *uniform* or *constant.*

145. Def. *Uniform Accelerating Force is measured by the velocity added (or subtracted) in a given time, as for instance, one second.*

Thus gravity, which during every second generates, in a body moving vertically downwards, a velocity of $32\frac{1}{5}$ feet, may be represented by this velocity (that is, by $32\frac{1}{5}$ feet); and then any other uniform force, as for instance, one which would generate a velocity of 1 foot in a second, will be measured by this its velocity, and its proportion to gravity will be that of 1 to $32\frac{1}{5}$, or 10 to 322.

146. Prop. *With uniform accelerating forces, the velocity generated in any time is equal to the product of numbers representing the force and the time.*

Let f be the accelerating force; then f is the velocity generated in one second. And since the force is uniform, f will also be the velocity added in the next second; and $2f$ will be the velocity at the end of 2 seconds. In the same manner, $3f$ will be the velocity at the end of 3 seconds; and, generally, tf will be the velocity at the end of t seconds. If v be the velocity, $v = tf$.

If we suppose the velocity generated in equal fractions of a second to be equal, this equation will also be true when t is a fractional or mixed number.

Cor. Since $v = tf$, $f = \dfrac{v}{t}$.

Hence, in uniform forces, the quotient of the velocity generated, by the time in which it is generated, is constant, and measures the force.

Thus if, as in Art. 134, a velocity of 8 feet be generated in 4 seconds, the accelerating force is $\frac{8}{4}$ or 2.

The velocity generated by gravity in one second is $32\frac{1}{6}$ feet. Hence the accelerating force of gravity is $32\frac{1}{6}$.

147. Prop. *With uniform accelerating forces, the space described from the beginning of the motion is as the square of the time.*

During the time t, the velocity of a body acted on by a force f begins from 0, and increases incessantly up to a certain finite magnitude v. It is manifest, therefore, that the space described in any portion of the time, with this increasing velocity, is less than the space which would have been described in the same portion of time, if the velocity had been, during the whole portion, as great as it is at the end of that portion. Let the time t be divided into n equal portions τ, τ, &c. so that $n\tau = t$. Then, by the last Article, we know the velocities at the end of each of these portions of time; and the space which would have been described if these velocities had been respectively continued uniform through each portion of time will be found, by Art. 133, by multiplying the velocity by the time in each portion. Thus, at the end of

the 1^{st}, 2^{d}, 3^{d}, 4^{th} ... n^{th} of the portions τ,

the velocities are $f\tau$, $2f\tau$, $3f\tau$, $4f\tau$,...$nf\tau$.

And if these velocities had been uniform through their respective times τ, the space described would have been

in the 1^{st}, 2^{d}, 3^{d}, 4^{th}......n^{th},

$f\tau^2$, $2f\tau^2$, $3f\tau^2$, $4f\tau^2$...$nf\tau^2$.

Now the sum of all these is

$$f\tau^2(1 + 2 + 3 + \dots n) = f\tau^2 \frac{n \cdot (n+1)}{2}$$

$$= \frac{f\tau^2 n^2}{2} + \frac{f\tau^2 n^2}{2n} = \frac{ft^2}{2} + \frac{ft^2}{2n}, \text{ because } \tau n = t.$$

Now the space which is actually described by the uniformly accelerated body is, as has been said, in each of these portions less than the corresponding space just found. Hence the

whole space described, which we call s, is less than the sum of all these spaces; that is, s is less than $\frac{ft^2}{2} + \frac{ft^2}{2n}$.

But the sum of all the spaces described with the uniform velocities will differ less and less from the actual space, as the portions of time are made smaller and smaller, and as their number is consequently made larger and larger; and by increasing this number indefinitely, the aggregate of the spaces described with the successive velocities, approaches indefinitely near to the space described with the accelerated motion; that is,

s approaches to $\frac{ft^2}{2} + \frac{ft^2}{2n}$, when n becomes indefinitely large.

Or $s = \frac{ft^2}{2}$, because the fraction $\frac{ft^2}{2n}$ becomes indefinitely small.

Hence s varies as t^2.

COR. 1. Since $s = \frac{1}{2}gt^2$, $t = \sqrt{\frac{2s}{g}}$.

COR. 2. In the two equations $v = ft$, $s = \frac{1}{2}ft^2$, we have, four quantities, any two of which serve to determine the other two. By elimination we obtain the following result;

$$v^2 = 2fs.$$

COR. 3. The space described in t seconds $= \frac{1}{2}ft^2$;

...... in $t - 1$ seconds $= \frac{1}{2}f(t-1)^2 = \frac{1}{2}f(t^2 - 2t + 1)$;

therefore, subtracting, we have

the space in the t^{th} second $= \frac{1}{2}f(2t - 1)$.

Hence the spaces in the 1st, 2d, 3d, 4th, &c. seconds are $\frac{1}{2}f.1$, $\frac{1}{2}f.3$, $\frac{1}{2}f.5$, $\frac{1}{2}f.7$, &c., and are as the odd numbers 1, 3, 5, 7, &c.

148. PROP. *The space described by a body uniformly accelerated from rest, is half the space described in the same time with the last acquired velocity.*

For by the last Proposition, $s = \frac{1}{2}tv$, and tv is the space described in the time t with the velocity v. (Art. 133.)

Cor. Hence also the space through which the body moves in the first second is the half of f, because f is the velocity acquired in $1''$.

149. Prop. *Let a body be projected with a given velocity* (u), *and acted on in the same direction by a constant force* (f); *the space described in a time* (t) *is equal to the space described with the velocity of projection,* plus *the space described from rest by the action of the force in the same time.*

If the body is, at a certain point, moving with a certain velocity, its motion after that point will be the same, however we suppose the velocity to have been acquired. Hence the motion will be the same, if we suppose that velocity to have been generated by the force accelerating the body from rest. Let the force f generate the velocity u by acting for a time t', through a space s'. Hence (Art. 146) $u = ft'$. Let the body afterwards continue to be acted on by the same force, and describe a space s in a time t; so as to describe a space $s' + s$ from rest in a time $t' + t$. Hence we have, by Art. 147,

$$s' + s = \tfrac{1}{2}f(t' + t)^2 = \tfrac{1}{2}f(t'^2 + 2t't + t^2),$$

$$\text{and,} \quad s' = \tfrac{1}{2}ft'^2;$$

$$\text{therefore,} \quad s = \tfrac{1}{2}f(2t't + t^2) = ft't + \tfrac{1}{2}ft^2;$$

$$\text{but,} \quad u = ft'; \quad \therefore \; s = tu + \tfrac{1}{2}ft^2.$$

Since tu is the space which the body would have described in the time t, with the uniform velocity u, and $\frac{1}{2}ft^2$ the space through which the force would have drawn it in the same time; it appears that the space thus described in any time is equal to the space described with the velocity of projection, *plus* the space described from rest by the action of the force.

Cor. By Art. 147, Cor. 2, $v^2 = 2f(s' + s)$; $u^2 = 2fs'$:

$$\text{hence } v^2 - u^2 = 2fs; \quad v^2 = u^2 + 2fs.$$

150. Prop. *When a body is projected in a direction opposite to that in which the force acts, the same formulæ will be true as in Article* 149, s *being the space, and* t *the time, from the end of the motion.*

In this case, the force will diminish the velocity; and, since the force is constant, will produce equal decrements in equal times. In a certain time, the body will be reduced to rest, and during this time, the velocity will go on decreasing by exactly the same degrees by which it increased when a body was accelerated from rest. Hence the spaces and times reckoned from the end of this motion will be the same as the spaces and times reckoned from the beginning of the former motion, and the same formulæ will be true.

Cor. 1. Let the body be projected with the velocity u, and let t' be the time and s' the space in which the whole of the velocity would be destroyed by the action of the force in the opposite direction. In a time t let a space s be described; then in the remaining time $t'-t$, from the end of the motion in a direction opposite to the force, there would be described a space $s'-s$: Hence we have

$$s' = \tfrac{1}{2}ft'^2;$$

$$s' - s = \tfrac{1}{2}f(t'-t)^2 = \tfrac{1}{2}f(t'^2 - 2t't + t^2);$$

$$\therefore s = \tfrac{1}{2}f(2t't - t^2) = ft't - \tfrac{1}{2}ft^2.$$

$$\text{and since } u = ft', \quad s = tu - \tfrac{1}{2}ft^2.$$

Cor. 2. Hence, as in the last Article, the space in any time is equal to the space described with the velocity of projection, *minus* the space described from rest by the action of the force in that time.

Cor. 3. Similarly, the velocity after any time is equal to the velocity of projection, *minus* the velocity generated by the force in that time.

SECTION III.

MOTION BY GRAVITY.

151. Prop. *Gravity is a uniform force.*

This Proposition is proved by induction from facts.

The facts which are included in the induction are such as the following:—

(1.) Bodies falling directly downwards fall quicker and quicker as they descend.

It was inferred, from a consideration of these cases, that the additions of velocity in the falling bodies are caused by gravity.

An attempt was made to assign the law of the increase of velocity conjecturally, by introducing the Definition, that a *uniform force* is a force which, acting in the direction of a body's motion, *adds equal velocities in equal spaces*, and the Proposition, that *gravity is a uniform force.*

The Definition is self-contradictory. But if it had not been so, the Proposition could only have been proved by experiment.

(2.) It appeared by experiment, that when bodies fall (down inclined planes) the spaces described are as the squares of the times from the beginning of the motion.

This was distinctly explained and rigorously deduced, by introducing the Definition of uniform force;—that it is a force which, acting in the direcion of the body's motion, *adds equal velocities in equal times;*

And the Principle,—that *gravity* (on inclined planes) *is a uniform force.*

For it has been proved, that this Definition being taken, the spaces described in consequence of the action of a uniform force are as the squares of the times from the beginning of the motion. And if the force be other than uniform, the spaces will not follow this law. Therefore the Proposition, that gravity on inclined planes is a uniform force, is the only one which will account for the results of experiment.

Also, if the force of gravity on all inclined planes be a uniform force, the force of gravity when bodies fall vertically is uniform; for when the inclined plane, from being nearly vertical, becomes quite vertical, the law must remain the same.

(3.) The Proposition is further confirmed, by shewing that its results, obtained deductively, agree with experiments made upon two bodies which draw each other over a fixed pully (Atwood's Machine); and—(4) with the observed times of oscillation of pendulums.

For the results of experiment in these cases agree with the consequences of the Proposition, that gravity is a uniform force.

Also it appears, that when gravity acts in a direction opposite to that of a body's motion, it *subtracts* equal velocities in equal times.

Therefore gravity is a uniform accelerating (or retarding) force.

152. Prop. *Gravity (considered as an accelerating force,) is the same in all bodies, whatever be the difference of material or magnitude.*

In bodies of different material this was proved by Newton from experiments upon pendulums: (Principia, Book III. Prop. 6:) he inferred from his observations that all substances would, by gravity, descend to the earth with equal velocities.

That bodies of the same material and different magnitudes would descend with the same velocity, is easily seen. For if one body be 10 times the other, let the first be divided into 10 bodies, each equal to the second. If these were all to fall at the same time from the same point, but separate, they would descend each with the same velocity as the second body. Hence if they were supposed to be connected and united, they would not accelerate or retard each other's motions; and therefore the whole mass would still descend with the same velocity.

The intensity of gravity, or the space through which a body would fall in 1″, must be determined by experiment. The most accurate observations for this purpose are those that are made upon pendulums. It will be shewn hereafter, that, knowing the length of a pendulum which oscillates once in a second, we can find the space through which a body would fall in the same time. By the latest experiments of this kind, it appears that in the latitude of London, and at the surface of the earth, a body would in vacuo fall through a space of 193.14 English inches, or $16\frac{1}{10}$ feet nearly. Consequently, (Art. 148) the velocity generated in that time would in the same time carry it through 386.28 inches, or $32\frac{1}{5}$ feet; and thus this space measures the velocity generated in 1″ by gravity, and is therefore the value of that force, according to the measure of accelerating forces. This quantity will generally be represented by g.

153. Hence we can easily solve all questions relating to the fall of bodies in vacuo by gravity. We have only to substitute the known quantity g for f in the formulæ of Art. 147: as is seen in the following examples.

Ex. 1. To find the *height due to* any given velocity.

(The height due to any velocity is the vertical space through which a body must fall by the force of gravity in order to acquire the velocity.)

The formula $v^2 = 2gs$, (Art. 147, Cor. 2) v and g being given, enables us to find s.

Thus, to find the height due to a velocity of 1000 feet per second,

$$s = \frac{v^2}{2g} = \frac{(1000)^2}{64.4} = 15528 \text{ feet.}$$

Ex. 2. To find how far a body will fall in vacuo in $2\frac{1}{2}''$, and the velocity acquired.

By the expression for s in Art. 147, putting g for f;

$$s = \tfrac{1}{2}gt^2 = \tfrac{1}{2} \times 32.2 \times \left(\frac{5}{2}\right)^2 \text{ feet} = 100.6 \text{ feet.}$$

By the expression for the velocity, Art. 146;

$$v = gt = 32.2 \times \frac{5}{2} \text{ feet} = 80.5 \text{ feet.}$$

Ex. 3. A body is projected upwards with a velocity of 100 feet; to find how high it will rise, and in what time it will reach its greatest height.

By Art. 150, the height to which it will rise will be the same as the height down which it must fall to acquire the velocity. Hence, by Art. 147, Cor. 2,

$$s = \frac{v^2}{2g} = \frac{(100)^2}{2 \times 32.2} = \frac{10000}{64.4} = 155.28 \text{ feet.}$$

Similarly, the time of the ascent is equal to the time of the descent; hence, by Art. 146,

$$t = \frac{v}{g} = \frac{100}{32.2} = 3.1''.$$

Ex. 4. On the same supposition, to find how high the body will ascend in $2''$.

Putting g for f in Art. 150, Cor. 1, we have

$$s = tu - \tfrac{1}{2}gt^2 = 2 \times 100 - \tfrac{1}{2} \times 32.2 \times 4 = 200 - 64.4 = 135.6 \text{ feet.}$$

By means of our formulæ we can easily solve such problems as the following:

Ex. 5. A person drops a stone into a well, and after t seconds hears it strike the water; to find the depth to the surface of the water.

We neglect the resistance of the air. The velocity of sound, as appears by experiment, is uniform, and equal to 1130 feet in a second. Now the time between dropping the stone and hearing the sound, is equal to the time of the stone falling the depth of the well, together with the time of the sound rising through the same distance. Let x be this depth, and n the velocity of sound. Then

$$\text{time of falling through } x = \sqrt{\frac{2x}{g}}; \text{ (Art. 147, Cor. 1.)}$$

$$\text{time of sound's passage} = \frac{x}{n};$$

$$\therefore \frac{x}{n} + \sqrt{\frac{2x}{g}} = t;$$

$$\therefore \frac{2x}{g} = t^2 - \frac{2tx}{n} + \frac{x^2}{n^2};$$

$$\therefore x^2 - 2\left(tn + \frac{n^2}{g}\right)x = -t^2n^2;$$

$$x = tn + \frac{n^2}{g} \pm \sqrt{\left(\frac{2tn^3}{g} + \frac{n^4}{g^2}\right)}.$$

The negative sign must be taken. Also it will be found that $\frac{n}{g} = 35$ nearly;

$$\therefore x = n\left\{t + \frac{n}{g} - \sqrt{\left(\frac{2tn}{g} + \frac{n^2}{g^2}\right)}\right\}$$

$$= n\left\{t + 35 - \sqrt{(70t + 1225)}\right\}.$$

Thus let the time t be $3''$; then

$$x = n\left\{38 - \sqrt{1435}\right\} = n\left\{38 - 37.88\right\}$$

$$= .12 \times n = 135.6 \text{ feet.}$$

SECTION IV.

VARIABLE FORCES.

154. When accelerating forces are variable, the reasonings respecting the motion must involve the conception of a *limit;* and the calculations will require some mathematical mode of applying this conception; for instance, the Differential Calculus.

By the fundamental principle of the Differential Calculus, the *differential coefficient* of s with regard to t, is the limit of the ratios of the increments of s and t; and is denoted thus, $\frac{ds}{dt}$; (Doct. Lim. B. v. Art. 5.)

PROP. *If* s *be the space described in a time* t *by a point acted upon in the direction of its motion by an accelerating force* f*; and if* v *be the velocity at any moment, we have*

$$v = \frac{ds}{dt}, \quad f = \frac{dv}{dt}.$$

It has already been shewn, Art. 135, that the velocity is the limit of the ratio of the increments of the space and time.

Also, in like manner, the accelerating force is greater as the increment of velocity in a given time is greater, and is measured by the ratio of the increment of the velocity to the increment of the time when this ratio is constant, as we have seen in Art. 146, Cor.

Therefore when this ratio is not constant, we measure the accelerating force by the limit to which this ratio tends when we diminish the time, for the same considerations as in the case of the velocity. Hence, by the definition of differential coefficients,

$$v = \frac{ds}{dt}, \quad f = \frac{dv}{d}.$$

COR. 1. If f be a function of s, t and v may also be considered as functions of s. Hence

$$f = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt}, \text{ (Doct. Lim. B. v. Art. 20.)}$$

$$\text{or } f = v\frac{dv}{ds}.$$

COR. 2. When f is a known function of s, we may, by integrating the equation just obtained, find the value of v^2, as a function of s: and hence the value of t as a function of s; for $\frac{dt}{ds} = \frac{1}{v}$,

which being integrated, gives t.

COR. 3. If x be the space to be described, we have

$$v = -\frac{dx}{dt}, \quad v\frac{dv}{dx} = -f; \quad \frac{dt}{dx} = -\frac{1}{v},$$

for the increment of s is always the same as the decrement of x, and hence the differential coefficients with regard to x and to s are the same quantities with opposite signs.

COR. 4. Integrating the equation $v\frac{dv}{dx} = -f$, in Cor. 3, we have

$$\tfrac{1}{2}v^2 = \text{constant} - \textstyle\int_x f; \quad \text{or } v^2 = C - 2\int_x f;$$

where C is the value of $2\int_x f$ when the motion begins, or when $v = 0$.

We shall not here pursue the subject of variable forces in general; but the case in which the force varies directly as the distance is one which will hereafter come under our notice, and we shall here apply to it the above formula.

155. PROP. *The accelerating force varies as the distance from a fixt point, and a body moves from rest at a distance* h, *in a line passing through this point: to determine the motion of the body.*

Let x be the distance from the fixt point; μx the force; and let the body move towards the point. Then by Art. 154, Cor. 3,

$$v\frac{dv}{dx} = -\mu x, \quad \tfrac{1}{2}v^2 = \mu h^2 - \mu x^2; \quad v = \sqrt{2\mu}\,.\sqrt{(h^2 - x^2)};$$

$$\frac{dt}{dx} = -\frac{1}{v} = -\frac{1}{\sqrt{2\mu}\,.\sqrt{(h^2 - x^2)}} : \text{ hence integrating,}$$

$$t = \frac{1}{\sqrt{2\mu}}\,\text{arc}\left(\cos = \frac{x}{h}\right),$$

which is 0 when $x = h$, that is, when the motion begins; as it should be.

To find the time, T, of the whole fall to the center, make $x = 0$, and we have $T = \frac{1}{\sqrt{2\mu}} \cdot \frac{\pi}{2}$.

If the earth were a homogeneous sphere consisting of attractive particles, the force of gravity at any point below the surface would be as the distance from the center. Hence we may apply our formulæ to this case.

Ex. On the above suppositions, find the time of a body falling down a vacant tube from the surface to the center of the earth.

Here, if h be the earth's radius, μh is the force of gravity at the earth's surface, or $\mu h = g$. Hence $\mu = \frac{g}{h}$, and

$$T = \sqrt{\frac{h}{2g}} \cdot \frac{\pi}{2}.$$

In this case $h = 20000000$ feet nearly; hence $T = 880''$ nearly, $= 14\frac{2}{3}$ minutes.

SECTION V.

THE SECOND LAW OF MOTION.

156. The Second Law of Motion. *When any force acts upon a body in motion, the motion which the force would produce in the body at rest is compounded with the previous motion of the body.*

The facts which this induction includes are, in the first place, such as the following:—

(1.) A stone dropped by a person in motion, is soon left behind.

From (1.) it was inferred that if the earth were in motion, bodies dropt or thrown would be left behind.

But it appeared that the stone was not left behind so long as it was moving in free space, and was only stopt when it came to the ground. Again, it was found by experiment,

(2.) That a stone dropt by a person in motion describes such a path that, relatively to him, it falls vertically.

(3.) A man throwing objects and catching them again uses the same effort whether he be at rest or in motion.

Again, such facts as the following were considered:—

(4.) A stone thrown horizontally or obliquely describes a bent path and comes to the ground.

It was at first supposed that the stone does not fall to the ground till the original velocity is expended. But when the First Law of Motion was understood, it was seen that the gravity of the stone must, from the first, produce a change in the motion, and deflect the stone from the line in which it was thrown. And by more exact examination it appeared that (making allowance for the resistance of the air),—(5.) the stone falls below the line of projection by exactly the space through which gravity in the same time would draw it from rest.

These facts were distinctly explained and rigorously deduced by introducing the Definition of Composition of Motions;—that two motions are *compounded* when each produces its separate effect in a direction parallel to itself;

And the Principle,—that when a force acts upon a body in motion, the motion which the force would produce in the body at rest is compounded with the previous motion of the body.

The Proposition is confirmed by shewing that its results, deduced by demonstration, agree with the facts.

SECTION VI.

PROJECTILES.

157. *A parabola* is a curve determined by the condition, that the distances of any point (P, Fig. Art. 160,) in the curve, from a fixt point (S) and from a straight line (DE) are equal ($SP = PD$). (Conic Sections.)

The fixt point is called the *Focus;* the straight line is called the *Directrix.*

158. Draw SE (Fig. Art. 160,) perpendicular to DE, PM perpendicular upon ES; bisect SE in A, and let $AE=AS=m$, $AM=x$, $MP=y$: then $y^2=4mx$. (See *Conics*; or thus:)

For $y^2=PM^2=SP^2-SM^2=PD^2-SM^2$ (Art. 157)

$$=EM^2-SM^2=(x+m)^2-(x-m)^2=4mx.$$

The quantity $4m$ is called the *principal parameter*; A is the vertex: AM is the *axis*.

Cor. Hence $x=\dfrac{y^2}{4m}$.

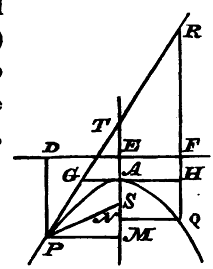
159. If PT be a tangent meeting the axis in T, $MT=2x$.

It is proved, *Conics*, Parab. Prop. v. that

$$MT=2AM, \text{ that is, } MT=2x.$$

160. If Q be any point in the curve, and if QR, parallel to the axis, meet the tangent PT in R, RQ varies as PR^2.

Draw QN perpendicular to the axis, and let $AN=u$, $QN=v$; therefore (Art. 158,) $v^2=4mu$. Also let GAH, perpendicular to the axis, meet PR in G and QR in H. Since $TA=\frac{1}{2}TM$, $GA=\frac{1}{2}PM=\frac{1}{2}y$; and $AH=NQ=v$, $\therefore GH=\frac{1}{2}y+v$. And $PM:MT::GH:HR$; that is,



$y : 2x :: \frac{1}{2}y+v : HR$; or since $x=\dfrac{y^2}{4m}$,

$$y:\frac{y^2}{2m}::\tfrac{1}{2}y+v:HR;\ \therefore HR=\frac{y^2+2yv}{4m};\ \text{also } HQ=u=\frac{v^2}{4m};$$

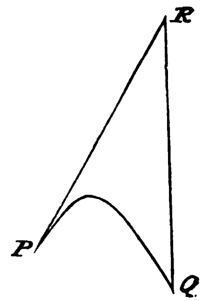
therefore $RQ=\dfrac{y^2+2yv+v^2}{4m}=\dfrac{(y+v)^2}{4m}=\dfrac{DF^2}{4m}$, if RQ meet DE in F. Hence, if the point P be fixt while Q moves, RQ varies as DF^2. Also DF varies as PR; therefore RQ varies as PR^2. (Also proved in *Conics*.)

161. Bodies projected or thrown in any manner and acted upon by gravity, are called *Projectiles*.

PROP. *A projectile describes a Parabola.*

We here leave out of our account the resistance of the air.

Let a body be projected from a point P in any direction PR; and let PR be the space which the body moving with the original velocity would describe in time t; and RQ the space through which it would fall in the same time by the force of gravity. Then the body will, by the second law of motion, (Art. 156) be found at the point Q at the end of the time. And if v be the velocity of projection, by Art. 146 and 147,



$$PR = tv; \quad RQ = \tfrac{1}{2}gt^2.$$

Hence PR^2 varies as t^2v^2, that is, as t^2; and RQ varies as $\frac{1}{2}gt^2$, that is, as t^2: therefore RQ varies as PR^2; and the curve is a parabola.

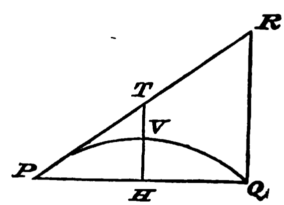
162. PROP. *To find the range on a horizontal plane, and the time of flight, of a body projected at a given elevation.*

The *Range* of a projectile on any plane is the distance from the point of projection, taken in the plane, to the point where the projectile strikes the plane again.

The *Time of flight* is the time employed in the motion from the instant of projection till the projectile strikes the plane.

The *Elevation* is the angle which the direction of projection makes with a horizontal plane.

Let PQ be the horizontal plane, PR the direction of projection, RQ vertical; the angle RPQ (the elevation) $= \alpha$; the velocity of projection $= v$; the time of flight $= t$; the range $= r$. As in the last Proposition,



$$PR = tv, \quad RQ = \tfrac{1}{2}gt^2.$$

$$\text{Hence } \sin\alpha = \frac{RQ}{PR} = \frac{\frac{1}{2}gt^2}{tv}: \text{ hence } t = \frac{2v \cdot \sin\alpha}{g}.$$

$$\text{And } r = PR\cos\alpha = tv \cdot \cos\alpha = \frac{2v^2\sin\alpha\cos\alpha}{g}.$$

But $2 \sin a \cos a = \sin 2a$. Hence $r = \frac{v^2}{g} \sin 2a$.

Ex. A body is projected at an elevation of 5^0 with a velocity of 1000 feet a second: to find the range and time of flight.

If we put 32 feet for g, $\frac{v^2}{g} = 31250$ feet.

Also $\sin 5^0 = \frac{1}{12}$, and $\sin 10^0 = \frac{1}{6}$, nearly. Hence $t = \frac{2v}{g} \sin a$ $= 5\frac{1}{5}$ seconds, nearly.

And $r = \frac{v^2}{g} \sin 2a = 5208\frac{1}{3}$ feet.

Cor. 1. The *greatest range* for a given velocity is when $\sin 2a$ is greatest, that is, when $2a$ is 90^0, or $a = 45^0$. In that case, the range is $\frac{v^2}{g}$.

Cor. 2. If h be the *height due* to the velocity v, $v^2 = 2gh$ (Art. 147). Hence the greatest range $= 2h$.

163. Prop. *To find the greatest height of a projectile above a horizontal plane.*

If H (Fig. Art. 162,) bisect PQ, and HVT drawn vertical meet the curve in V and the tangent PR in T, V is the vertex of the curve (*Conics*); and HV is the greatest height. In that case, HV is $\frac{1}{2}HT$, and HT is $\frac{1}{2}QR$.

$$\text{Hence } HV = \tfrac{1}{4}RQ = \tfrac{1}{8}gt^2;$$

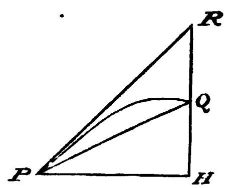
$$\text{or, greatest height} = \frac{1}{8} g \left(\frac{2v \cdot \sin a}{g}\right)^2 = \frac{v^2}{2g} \sin^2 a.$$

Taking the same example as in the last Article,

$$\text{greatest height, } 15625 \times \frac{1}{12^2} = 108\tfrac{1}{2} \text{ feet nearly.}$$

164. Prop. *To find the range upon an inclined plane, and time of flight, of a projectile.*

Let, as before, v be the velocity, t the time; the range $r = PQ$; the elevation of the projection $RPH = a$; the inclination of the plane to the horizon, $QPH = \iota$.



Hence $RPQ = \alpha - \iota$.

Then, RQ being vertical, we have $PR = tv$, $RQ = \frac{1}{2}gt^2$.

Also, $\dfrac{RQ}{PR} = \dfrac{\sin RPQ}{\sin PQR} = \dfrac{\sin RPQ}{\sin PQH}$

$= \dfrac{\sin RPQ}{\cos QPH}$; or $\dfrac{\frac{1}{2}gt^2}{tv} = \dfrac{\sin(\alpha - \iota)}{\cos\iota}$; whence $t = \dfrac{2v}{g}\,\dfrac{(\sin\alpha - \iota)}{\cos\iota}$.

$$PH = PR\,.\cos RPH = tv\cos\alpha = \frac{2v^2}{g}\,\frac{\sin(\alpha - \iota)\cos\alpha}{\cos\iota}.$$

$$PQ = \frac{PH}{\cos\iota};\ \text{hence}\ r = \frac{2v^2}{g}\,\frac{\sin(\alpha - \iota)\cos\alpha}{\cos^2\iota}.$$

Cor. 1. By Trigonometry,

$$2\sin(\alpha - \iota)\cos\alpha = \sin\{\alpha + (\alpha - \iota)\} - \sin\{\alpha - (\alpha - \iota)\}$$
$$= \sin(2\alpha - \iota) - \sin\iota.$$

$$\text{Hence}\ r = \frac{v^2}{g\cos^2\iota}\{\sin(2\alpha - \iota) - \sin\iota\}.$$

Cor. 2. Hence the greatest range will take place when $\sin(2\alpha - \iota)$ is greatest (for $\sin\iota$ is constant); that is, when $2\alpha - \iota$ is a right angle,

$$2\alpha - \iota = 90^0,\quad \alpha = 45^0 + \frac{\iota}{2}.$$

165. Prop. *If two bodies be projected at equal angles above and below the angle of the greatest range of the projectile on an inclined plane, the ranges of the two bodies on the inclined plane will be equal.*

Let $45^0 + \frac{\iota}{2} \pm \theta$ be the elevations of the projections: then

$$\alpha = 45^0 + \frac{\iota}{2} \pm \theta;\quad 2\alpha - \iota = 90^0 \pm \theta;$$

$$\sin(2\alpha - \iota) = \sin(90^0 \pm \theta).$$

Now $\sin(90^\circ+\theta)$, and $\sin(90^\circ-\theta)$, and each $=\cos\theta$.

Hence for both cases, $r = \dfrac{v^2}{g\cos^2\iota}(\cos\theta - \sin\iota)$.

166. Prop. *The motions of bodies projected with great velocities in air will differ much from their parabolic motions which would take place in a vacuum.*

It appears, from reasoning and experiment, that when a plane moves perpendicularly to itself in a fluid, the resistance to its motion is equal to the weight of a column of fluid, having for its base the area of the plane, and for its height the vertical height due to the velocity of the plane's motion. Thus a plane whose area is a square inch, moving in air with a velocity v in a direction perpendicular to the plane, would experience a resistance equal to a column of air of one square inch base, and of the height $\dfrac{v^2}{2g}$. This height, for a velocity of 1000 feet a second, would be 15528 feet: and the content of the column, 108 cubic feet. Now the specific gravity of air being about $\dfrac{1}{840}$ of that of water, and the weight of a cubic foot of water being 1000 ounces avoirdupoise, the weight of a cubic foot of air is $1\frac{1}{5}$ ounce, and the weight of the column of air is 130 ounces, which is the perpendicular resistance on any plane whose area is a square inch. And the resistance on a circle whose diameter is an inch, is about $\frac{4}{5}$ the above resistance; i. e. it is nearly 104 ounces.

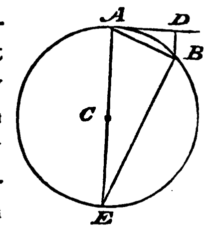
But the resistance on a sphere, it appears by calculation, is one half the resistance on a circle of equal radius. Hence the resistance on a sphere whose diameter is an inch, is 52 oz.

The weight of a leaden ball of an inch diameter is about 3.4 oz. Hence the resistance is about fifteen times the force of gravity. This resistance acts in the direction of a tangent to the curve, and retards the body's motion. In one second it will bring the ball nearly 15×16 feet, or 240 feet, behind its parabolic place. Hence in such cases the path will deviate much from a parabola.

SECTION VII.

CENTRIFUGAL FORCE.

167. If a body is made to describe a circle with a uniform velocity, it must be acted upon by a force tending towards the center of the circle. For at any point A, of the motion in AB, the body has a tendency to go forwards in the direction AD, the tangent to the circle, by the first law of motion. And in order that it may be prevented from doing this, it must be acted upon



by a force transverse to its motion, which may deflect it from the tangent. Such a force, continually acting, will cause it to describe a curve: and the deflection from the tangent in a given small time will be the effect of the deflecting force in magnitude and direction, by the second law of motion. If the deflecting force were constant in magnitude and direction, its effect in any finite time would be compounded with the previous motion of the body, as in the last section. But if the direction or magnitude, or both, vary, the deflection will be the effect of the force, *taking the limit.*

168. Prop. *When a body describes a circle with a uniform velocity, and is retained in its path by a force tending to the center, this force is represented by the square of the velocity divided by the radius.*

Let v be the velocity, r the radius, f the force which acts towards the center. Let τ be the small time in which the body, not acted upon by the central force, would describe the small portion AD of the tangent; and let DB be the deflection by which the body is brought to B. Hence, at the limit,

$$AD = v\tau, \quad BD = \tfrac{1}{2}f\tau^2.$$

But if AE be the diameter, the triangles EAB, ABD are similar; for the angles ADB, ABE are right angles, and EAB, ABD are equal.

Hence $EA : AB :: AB : BD$; therefore $EA \times BD = AB \times AB$; and at the limit, $BD = \frac{1}{2}ft^2$, $AB = v\tau$;

$$\text{hence } 2r \times \tfrac{1}{2}f\tau^2 = (v\tau)^2. \text{ Therefore } f = \frac{v^2}{r}.$$

Cor. 1. If ω be the *angular velocity*, (the angle described in a unit of time,) $r\omega$ will be the arc described in a unit of time, that is, the linear velocity. Hence $v = r\omega$, and $f = r\omega^2$.

Cor. 2. If T be the *periodic time*, (the time of describing the whole circumference of the circle,) we have $Tv = 2\pi r$.

$$\text{Hence } v = \frac{2\pi r}{T}; \text{ and } f = \frac{4\pi^2 r}{T^2}.$$

Hence we may solve such questions as the following.

Ex. A body revolves uniformly in a circle of which the radius is 3 feet, making a revolution in each second: to find the central force. Here $r = 3$, $T = 1$; hence

$f = \dfrac{4\pi^2 r}{T} = 118$, the force of gravity being 32, nearly.

169. When a body moves in a circle, the force which urges it towards the center and compels it to describe the curve, is called the *central* or *centripetal* force: the tendency from the center, which makes the centripetal force necessary, is called the *centrifugal* force. These two forces measure each other. If the centripetal force cease to act, the body flies off from the center, and this result is ascribed to centrifugal force. Thus when a stone at the end of a string is whirled so fast that the string breaks, the string is broken by the centrifugal force. In every circular motion there is centrifugal force, and the amount of it may be found by the previous formula for centripetal force.

Ex. To compare the centrifugal force which a body at the Equator exerts in virtue of the earth's diurnal rotation, with the force of gravity.

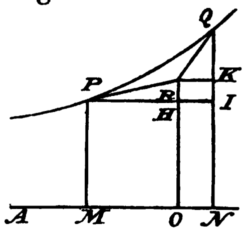
Here $r = 20000000$ feet, $T = 86000$ seconds. Hence

$f = \dfrac{4\pi^2 r}{T^2} = \dfrac{1}{9}$ nearly, and $g = 32$: hence $f : g :: 1 : 288$ nearly.

SECTION VIII.

GENERAL FORMULÆ.

170. Prop. *To find the equations of motion of a body moving in a plane and acted upon by any forces in that plane.*



Let the body come to the point P, moving in the direction PR, with the velocity V; and at P, let it be acted on by forces X, Y in the directions of the co-ordinates x, y: and after a small time h, let the body be at Q. Let PR be the space which the body would have described in the time h if no forces X, Y had acted: then the space RQ is that due to the action of the forces X, Y; and would be equal to the space which the point would describe by the action of the forces X, Y, if those forces were constant and parallel during the time h: but this not being the case generally, RQ will be equal to the space so described *at the limit*. Let AM, MP be rectangular co-ordinates (x, y) of P; AN, NQ the co-ordinates of Q; AO, OR the co-ordinates of R; and resolving RQ into RK, KQ in the direction of the co-ordinates x, y, RK is the space which the point would describe by the action of the force X, and KQ the space which the point would describe by the action of the force Y in the time h, at the limit.

Let t be the time of arriving at P, measured from a given moment; then $t + h$ is the time of arriving at Q, and x, y are functions of t; hence, by Taylor's Theorem,

$$\text{for the point } Q,\ AN = x + \frac{dx}{dt}h + \frac{d^2x}{dt^2}\frac{h^2}{1.2} + \&c.$$

$$NQ = y + \frac{dy}{dt}h + \frac{d^2y}{dt^2}\frac{h^2}{1.2} + \&c.$$

$$\text{Hence } AN - x = AN - AM = MN = \frac{dx}{dt}h + \frac{d^2x}{dt^2}\frac{h^2}{1.2} + \&c.$$

$$NQ - y = NQ - NI = IQ = \frac{dy}{dt}h + \frac{d^2y}{dt^2}\frac{h^2}{1.2} + \&c.$$

Also $PR = Vh$; and if the angle $RPH = \theta$,

$$PH = Vh\cos\theta,\ HR = Vh\sin\theta.$$

Hence $RK = MN - MO = MN - PH$

$$= \left(\frac{dx}{dt} - V\cos\theta\right)h + \frac{d^2x}{dt^2}\frac{h^2}{1.2} + \&c.$$

$$KQ = IQ - IK = IQ - HR$$

$$= \left(\frac{dy}{dt} - V\sin\theta\right)h + \frac{d^2y}{dt^2}\frac{h^2}{1.2} + \&c.$$

But if X and Y were constant, we should have

$$RK = \tfrac{1}{2}Xh^2,\ KQ = \tfrac{1}{2}Yh^2.$$

Hence, at the limit,

$$\left(\frac{dx}{dt} - V\cos\theta\right)h + \frac{d^2x}{dt^2}\frac{h^2}{1.2} + \&c. = \tfrac{1}{2}Xh^2,$$

$$\left(\frac{dy}{dt} - V\sin\theta\right)h + \frac{d^2y}{dt^2}\frac{h^2}{1.2} + \&c. = \tfrac{1}{2}Yh^2.$$

But these cannot be equal at the limit except the coefficient of h vanish, and the coefficients of h^2 be equal. Therefore

$$\frac{dx}{dt} = V\cos\theta,\quad \frac{dy}{dt} = V\sin\theta,$$

$$\frac{d^2x}{dt^2} = X,\qquad \frac{d^2y}{dt^2} = Y,$$

where X and Y are positive, when they tend to increase x and y, and t is the independent variable.

Cor. It is clear that if we had referred the path of the body to three rectangular co-ordinates, x, y, z, and if we had made X, Y, Z represent the whole forces in the directions of these co-ordinates, we should have had, by reasoning exactly similar,

$$\frac{d^2x}{dt^2} = X,\ \frac{d^2y}{dt^2} = Y,\ \frac{d^2z}{dt^2} = Z.$$

CHAPTER II.

MOVING FORCE.

SECTION I.

THE THIRD LAW OF MOTION.

171. It has already been said that we may have various measures of force according to the effect of force which we consider. In the former chapter we considered the effect of force in producing velocity without any regard to the quantity of matter moved: in this chapter we consider the mass moved, as well as the velocity communicated.

Def. I. *The momentum of a body in motion is proportional to its velocity and quantity of matter jointly.*

If the velocity and the quantity of matter of a body be both expressed in numbers, the momentum of the body will be expressed by the *product* of these numbers.

Def. II. *Moving force is force measured by the momentum which, in a given time, it would produce in a body.*

We have seen that force, as for instance, pressure, may produce velocity in a body at rest, and add velocity to a body in motion, so that the velocity may go on increasing with the time, in virtue of the continued action of the force. Now greater pressure is requisite to produce a large velocity than to produce a small velocity in the same time: and also a greater pressure is requisite to produce the same velocity in a large body than in a small one. Hence it appears that force, for certain purposes, may be conceived to depend both upon the velocity and upon the quantity of matter: and some of these purposes lead us to the above Definition.

The same force which generates momentum may also destroy it: hence moving force is force measured by the momentum which it generates or destroys in a given time.

Cor. If in two cases the moving force which generates or destroys motion be equal, the momentum generated or destroyed in any given time will also be equal.

172. We have to conceive pressures producing motions; and in order to reason concerning such cases, we must lay down certain axioms.

Ax. I. Equal pressures acting in the same manner for the same time upon equal bodies produce the same velocity.

Ax. II. If any system of equal detached material points, having its different particles acted upon by equal parallel forces, move parallel to itself and to the forces, the particles may be supposed to be rigidly connected, and to be acted upon by the same forces, and the motions will not be altered.

Ax. III. On the same suppositions, the parallel forces may be supposed to be added together so as to become one force, and the motions will not be altered.

173. Prop. *If two pressures* P, Q *act on two bodies of which the quantities of matter are* M, N, *and if* P : Q :: M : N, *the velocities produced in equal times will be equal.*

Let the pressure P be divided into M parts, each of which will be $\frac{P}{M}$; and let the body M be divided into M units. Then if a force $\frac{P}{M}$ act upon each of these units, in a parallel direction to P, and produce a velocity v in a time t, by Ax. I. the velocities will all be equal. But the M units may all be supposed to be rigidly connected, by Ax. II. (Art. 172), and the M forces $\frac{P}{M}$ may be supposed to be added together, by Ax. III., without altering the motion. Therefore when the force P acts upon the mass M, it will still generate a velocity v in a time t.

Now $\frac{Q}{N} = \frac{P}{M}$; hence the force $\frac{Q}{N}$, acting upon a unit of matter will still generate a velocity v in a time t. And hence, by the same reasoning as before, a force Q will generate the same velocity in N units connected; that is, in a body of which the quantity of matter is N. Therefore, in a time t, P will generate the same velocity in M as Q in N.

174. The velocity generated in a given body by any pressure acting for a given time is as the pressure; but this may be expressed otherwise, as follows:

The third law of motion. *When pressure generates or destroys motion in a given body, the accelerating force is as the pressure.*

This Proposition was collected by induction from facts.

The facts included in this induction are such as the following:—

(1) When pressure produces motion, the velocity produced is greater when the pressure is greater.

In order to determine in what proportion the velocity increases with the pressure, further consideration and inquiry are necessary.

It appeared that,

(2) On an inclined plane the velocity acquired by falling down the plane is the same as that acquired by falling freely down the vertical height of the plane (Galileo's experiment).

(3) When two bodies P, Q hang over a fixed pully, the heavier, P, descends, and the velocity generated in a given time is as $P - Q$ directly, and as $P + Q$ inversely (Atwood's Machine).

(4) The small oscillations of pendulums are performed in times which are as the square roots of the lengths of the pendulums.

(5) In the impacts of bodies the momentum gained by the one body is equal to the momentum lost by the other (Newton's Experiments).

(6) In the mutual attractions of bodies the center of gravity remains at rest.

These results are distinctly explained and rigorously deduced by introducing the Definition of uniform Accelerating Force;—that it is as the velocity generated (or destroyed) in a given time;

And the Principle—that the accelerating force for a given body is as the pressure.

Most of these consequences will be proved in the succeeding Propositions, (Articles 175, 176, 185, 187,) and thus this inductive Proposition is confirmed.

175. PROP. *When pressure generates or destroys motion, the moving force is as the pressure.*

Let two pressures P, Q act for the same time upon two bodies, of which the quantities of matter are M, N. Let $M : N :: P : X$, therefore (Art. 173) the force X would produce in N the same velocity in a given time which P produces in M. Therefore the accelerating force, when X acts on N, is the same as the accelerating force when P acts on M. But by Art. 174, the accelerating force when Q acts on N is to the accelerating force when X acts on N as Q to X; that is, as Q to $\frac{PN}{M}$. Whence, accelerating force of Q on N : accelerating force of P on $M :: Q : \frac{PN}{M} :: \frac{Q}{N} : \frac{P}{M}$. But since the accelerating forces retain a constant ratio, and act for the same time, the velocities which they generate are as the forces P, Q: (Art. 144.) Let these velocities be u, v; therefore

$$v : u :: \frac{Q}{N} : \frac{P}{M}; \text{ or } Nv : Mu :: Q : P.$$

Therefore the momenta generated in the given time are as the pressures $Q : P$; therefore, by the definition of moving force, Art. 171, the moving forces are in this proportion.

And the same is true of momenta destroyed when the pressures act in the opposite directions.

176. The action of two bodies is *direct*, when it takes place entirely in the line in which they are both moving.

PROP. *In the direct action of two bodies, the momenta gained and lost are equal, during any time.*

Let P act upon Q, and let X be the pressure which each exerts upon the other at any instant; for these pressures must be equal by Art. 13. Therefore the moving forces which act upon P and Q in opposite directions are each equal to X. Therefore by Article 171, the momentum generated in one

body, and that destroyed in the other, in the same time, are equal: that is, the momentum gained by the one body and that lost by the other body are equal.

This Proposition applies to mutual attraction, to mutual pressure, and to impact, which is a short and rapid pressure.

177. In the impact of bodies, their motions are determined by their being elastic or inelastic.

Elastic bodies are bodies which, when one impinges on another, separate with a velocity determined by the velocity with which, in the impact, they approach.

Perfectly elastic bodies are those which, after the impact, separate with a velocity equal to that with which they met in the impact. This is sometimes expressed by saying, that in *perfectly elastic bodies the force of restitution is equal to the force of compression.*

178. *Imperfectly elastic* bodies are those which separate after the impact with a velocity less than that with which they meet in the impact.

It appears by experiment (Newton's experiments) that the velocity with which imperfectly elastic bodies separate have, for each material, a constant ratio (constant for different magnitudes and velocities, different for different materials) to the velocity with which they approach in the impact. This is sometimes expressed by saying, that *in imperfectly elastic bodies the force of restitution bears a constant ratio to the force of compression.*

Inelastic bodies are those which have no tendency to separate, arising from the resistance of the material in impact.

179. Prop. *In the direct impact of two perfectly elastic bodies,* P, Q, *given* a, b, *the velocities before impact, to find* u, v, *the velocities after impact.*

Here a, b are supposed to be measured in the same direction, a being the greater; therefore $a - b$ is the velocity with which P approaches Q; also $v - u$ is the velocity with which Q separates from P after the impact. Therefore (Art. 177) $v - u = a - b$.

Also since the momentum lost by P in the impact is equal to that gained by Q, (Art. 176,) the total momentum is the same before as after the impact. Therefore

$$Pu + Qv = Pa + Qb.$$

And by the former equation, $Qv - Qu = Qa - Qb$.

Subtracting, $(P + Q)u = (P - Q)a + 2Qb$.

In like manner, $Pv - Pu = Pa - Pb$, and adding,

$$(P + Q)v = 2Pa - (P - Q)b.$$

$$\text{Hence } u = \frac{(P - Q)a + 2Qb}{P + Q}, \quad v = \frac{2Pa - (P - Q)b}{P + Q}.$$

Cor. 1. Let the two bodies be equal; then $P - Q = 0$, $u = b$, $v = a$.

The bodies interchange velocities.

Cor. 2. Let P impinge upon Q at rest; then $b = 0$.

$$u = \frac{P - Q}{P + Q}a, \quad v = \frac{2P}{P + Q}a.$$

Cor. 3. Since $Pa + Qb = Pu + Qv$,

$$P(a - u) = Q(v - b).$$

Also $v - u = a - b$, or $a + u = v + b$.

$$\text{Whence } P(a^2 - u^2) = Q(v^2 - b^2),$$

$$\text{or } Pa^2 + Qb^2 = Pu^2 + Qv^2.$$

180. Prop. *In the direct impact of imperfectly elastic bodies, the ratio of the force of restitution to the force of compression being given, the velocities after impact may be found.*

For example, let the force of restitution be half the force of compression; then $v - u = \frac{1}{2}(a - b)$; also $Pu + Qv = Pa + Qb$. Hence $Qv - Qu = \frac{1}{2}(Qa - Qb)$; $Pv - Pu = \frac{1}{2}(Pa - Pb)$; whence subtracting and adding

$$(P + Q)u = (P - \tfrac{1}{2}Q)a + \frac{3Q}{2}b.$$

$$(P + Q)v = \frac{3P}{2}a + (Q - \tfrac{1}{2}P)b.$$

Cor. Hence if $Q = P$, $u = \dfrac{a + 3b}{4}$, $v = \dfrac{3a + b}{4}$.

181. Prop. *In the direct impact of inelastic bodies, the velocity after impact is common to the two, and is* $\frac{Pa + Qb}{P + Q}$.

For if u be the velocity, $Pu + Qu$ is the momentum after impact. Therefore $Pu + Qu = Pa + Qb$, and $u = \frac{Pa + Qb}{P + Q}$.

182. Prop. *The motion of the center of gravity of two bodies is not affected by their direct impact.*

Let P, Q be two bodies moving in the same straight line, with the velocities a, b before impact, and u, v after impact.

At the moment of impact, let the distances of the centers of gravity P and Q from a fixt point in the line of the motion be h, k. Then, at any time previous to this moment by t seconds, the distances of the centers of P and Q from the fixt point will be $h - at$ and $k - bt$; and at any time t after the impact the distances will be $h + ut$, $k + vt$. At the former instant, the distance of the center of gravity from the fixt point will be

$$\frac{P(h - at) + Q(k - bt)}{P + Q}, \text{ or } \frac{Ph + Qk - (Pa + Qb)t}{P + Q}.$$

At the latter instant, the distance of the center of gravity from the fixt point will be

$$\frac{P(h + ut) + Q(k + vt)}{P + Q}, \text{ or } \frac{Ph + Qk + (Pu + Qv)t}{P + Q}.$$

At the instant of impact, the distance of the center of gravity from the fixt point is $\frac{Ph + Qk}{P + Q}$.

Therefore the space described along the line of motion, by the center of gravity, during the time t before the impact, is $\frac{(Pa + Qb)t}{P + Q}$; and after the impact is $\frac{(Pu + Qv)t}{P + Q}$; and these are equal, because $Pa + Qb = Pu + Qv$. Therefore the velocity of the center of gravity is the same before and after the impact.

Cor. Hence if the center of gravity of two bodies be originally at rest, it cannot be put in motion by the impact.

SECTION II.

CONSTANT MOVING FORCES.

183. Prop. *The accelerating force on any body is equal to the moving force divided by the quantity of matter.*

Let a moving force P produce in a mass M, a velocity V in one second, and let F be the accelerating force. Then the moving force P is measured by the momentum MV, (Art. 171,) and the accelerating force F is measured by the velocity V: (Art. 145) therefore $F = \frac{P}{M}$.

Cor. 1. If the moving force, P, be the weight of the mass M, the accelerating force, F, is gravity, which we represent by g; ($g = 32$ feet nearly). Hence $g = \frac{P}{M}$, and $P = Mg$.

Cor. 2. Hence if P, the weight, be known, the quantity of matter is $= \frac{P}{g}$.

Cor. 3. If a weight P move a weight Q by its pressure, the accelerating force $= P \div \frac{Q}{g} = \frac{Pg}{Q}$.

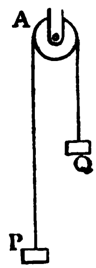
184. When a moving force acts on a mass and produces motion, the accelerating force is equal to a fraction which has the quantity of matter for its denominator. The quantity of matter thus involved is termed the *Inertia*. It is conceived as obstructing or resisting the action of the moving force; for the motion produced is less in proportion as this inertia is greater.

When the moving force acts on the mass *directly*, the denominator of the accelerating force is the mass simply. In this case the inertia may be termed *Simple Inertia*, to distinguish it from *Rotatory Inertia*, of which we shall afterwards have to speak.

185. In this and the three following Sections, when we speak of a body, or a heavy body, without further definition, we mean a mere material point. When the moving force which acts directly on a mass is constant, the accelerating

force is constant, and the formula for constant accelerating forces may be applied to determine the motion.

PROP. *Two heavy bodies of weights* P, Q, *hang over a fixt pully; to find the accelerating force.*



Let X be the tension of the string; therefore $P - X$ is the pressure on P downwards, and $\dfrac{P-X}{P}g$ the accelerating force (Art. 183, Cor. 3). Also $X - Q$ is the pressure on Q upwards, and $\dfrac{X-Q}{Q}g$ the accelerating force. And these accelerating forces are equal, for the velocities of the two bodies are equal at every instant. Therefore

$$\frac{P-X}{P}g = \frac{X-Q}{Q}g; \quad PQ - QX = PX - PQ; \quad X = \frac{2PQ}{P+Q}.$$

Hence $\dfrac{P-X}{P}g = \dfrac{P-Q}{P+Q}g$, the accelerating force.

Ex. Let $P = 41$, $Q = 39$: find the space described from rest in five seconds. The accelerating force $= \dfrac{2g}{80} = \dfrac{4}{5}$, if $g = 32$.

Hence the formula $s = \frac{1}{2}ft^2$ (Art. 147,) making $f = \dfrac{4}{5}$, $t = 5$, gives $s = 10$ feet.

186. PROP. *A body moving with a given velocity is reduced to rest by a certain constant resistance acting through a given space; to find the resistance.*

Let Q be the weight of the body, P the resistance, V the velocity, S the space in which the velocity is extinguished. The accelerating, that is, the retarding force, is $\dfrac{Pg}{Q}$, (Art. 183, Cor. 3;) and since the formulæ $v^2 = 2fs$ is applicable in this case, we have

$$V^2 = \frac{2PSg}{Q}; \quad P = \frac{QV^2}{2Sg}.$$

Ex. 1. Let a body, of weight 1, impinge into a soft mass of uniform resistance with a velocity 32; and lose all its velocity in penetrating 1 foot: to find the resistance.

$$P = \frac{(32)^2}{2 \cdot 32} = 16,$$

the resistance is 16 times the weight of the body.

Ex. 2. A ball, of weight 1, impinges on a block of wood with a velocity of 1600 feet a second, and loses all its velocity in penetrating 1 foot: in what time is it reduced to rest?

$$S = \tfrac{1}{2} t V: \text{ whence } t = \frac{2S}{V} = \frac{1}{800} \text{ of a second.}$$

Ex. 3. In this case, how much velocity does the ball lose in penetrating $\frac{1}{n}$ of a foot?

If v be the velocity after penetrating $\frac{1}{n}$ of a foot, when $\frac{n-1}{n}$ of a foot still remains to penetrate: since $v^2 = 2fs$, v^2 is as s, the space to be described; (Art. 150,) whence $V^2 : v^2 :: 1 : \frac{n-1}{n}$;

$$v^2 = \left(\frac{n-1}{n}\right) V^2 : V^2 - v^2 = \frac{1}{n} V^2.$$

The square of the velocity loses $\frac{1}{n}$ of its magnitude.

If n be very large, $v = V\sqrt{\left(1 - \frac{1}{n}\right)} = V\left(1 - \frac{1}{2n}\right)$ nearly; the velocity loses nearly $\frac{1}{2n}$.

Thus if n be 100, the velocity in the last example loses $\frac{1}{200}$ of its magnitude, and becomes 1592.

Ex. 4. When the ball passes through a thin slice of the resisting material, to find the momentum communicated to the slice.

Let the thickness of the slice be $\frac{S}{n}$, where S is the space necessary to extinguish the whole velocity: v the velocity after penetrating $\frac{S}{n}$: then, as above, $v^2 = \frac{n-1}{n} V^2$, $v = V\left(1 - \frac{1}{2n}\right)$ nearly, $V - v = \frac{V}{2n}$. Hence the momentum lost by the ball B is $\frac{BV}{2n}$; and this is the momentum gained by the slice.

If as above, $V = 1600$, and the thickness be $\frac{1}{100}$ of S, $n = 100$, and the momentum lost is $8B$; and this is the momentum gained by the slice.

If the inertia of the slice be C, the velocity acquired by the slice is $\frac{BV}{2nC}$. Thus if $C = 100B$ and $n = 100$, the velocity acquired by the slice, when the ball thus passes through it, is $\frac{V}{20000}$, or 1 inch per second, in the above case.

SECTION III.

MOTION ON INCLINED PLANES.

187. Prop. *The velocity acquired by a body falling from rest down an inclined plane is equal to the velocity acquired by a body falling freely through the same vertical height.*

The force which must act upwards along an inclined plane in order to support a body Q, is $\frac{QH}{L}$, H being the height and L the length of the inclined plane (Art. 61): therefore if the body is not supported, the force which urges it downwards is $\frac{QH}{L}$. This is the moving force; and the weight moved is Q: therefore (Art. 183, Cor. 3) the accelerating force is $\frac{Hg}{L}$. Also in the formula $v^2 = 2fs$, we must put L for s, because the whole length of the plane is the space described: therefore

$v^2 = 2\frac{Hg}{L}L = 2Hg$. And this is the same expression which we have when v is the velocity of a body falling freely through the height H.

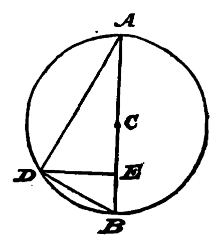
COR. 1. The velocities acquired by falling down all inclined planes, having a constant vertical height, are equal.

COR. 2. If at any point of an inclined plane the velocity be u, and if v be the velocity after descending through a vertical height h below that point, $v^2 = u^2 + 2gh$. For if h' be the height due to the velocity u, $h + h'$ will be the height due to the velocity v; whence by the Proposition, $u^2 = 2gh'$, $v^2 = 2g(h' + h) = u^2 + 2gh$.

188. PROP. *The time of falling from rest down the inclined plane is to the time of falling freely down the vertical height as the length of the plane to its height.*

If t be the time of falling down the inclined plane, since $s = \frac{1}{2}ft^2$, and as in last Article, $f = \frac{Hg}{L}$, we have $L = \frac{1}{2}\frac{Hg}{L}t^2$: whence $t^2 = \frac{2L^2}{Hg}$. Also if t' be the time of falling vertically down H, $t'^2 = \frac{2H}{g}$. Hence $t^2 : t'^2 :: \frac{2L^2}{Hg} : \frac{2H}{g} :: L^2 : H^2$; wherefore $t : t' :: L : H$.

189. PROP. *The time of falling down the vertical diameter of a circle by gravity is equal to the time of falling down any chord drawn through the highest (or lowest) point.*



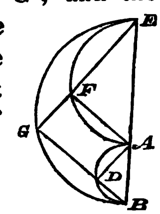
Let AB be the vertical diameter of a circle, AD any chord. Draw DE horizontal, and let $AE = H$, $AD = L$. Then, as above, if t = time down AD, $t^2 = \frac{2L^2}{gH}$. But by similar triangles, $EA : AD :: AD : AB$; that is, $H : L :: L : AB = \frac{L^2}{H}$. And if t' = time down AB, by Art. 147,

$$t'^2 = \frac{2AB}{g} = \frac{2L^2}{gH}. \quad \text{Hence } t' = t.$$

And the same proof will apply to the chord BD. Therefore, &c.

Cor. If two circles touch each other at the highest or lowest point, and any chord be drawn through this point, the time of falling from rest down the portion of the chord intercepted between the two circles is the same for all such chords.

Let the circles meet the chord EFG in F, G; and the vertical diameter EAB in AB. On AB describe a semicircle, and join BG meeting the semicircle in D, and join AD, AF. It is easily shown that FG is equal and parallel to AD; hence time of falling down FG = time down AD = time down AB; and is the same for all the chords.



Problems.

190. By aid of the Propositions just proved we may solve the following Problems by means of geometrical constructions.

It is required to find the plane of shortest descent;

Prob. I. From a given point P to a given straight line AB.

From the given point P draw a horizontal line meeting the given line in A. Take AQ downwards along the given line, and equal to AP: PQ will be the plane required.

Prob. II. From a given straight line AB to a given point P.

Draw PA as before; and along the given line measure a distance AR upwards from A, equal to AP; the line RP joining the extremity of this distance with the point P, is the plane required.

Prob. III. From a given point without a given circle to the circle.

Join the given point with the lowest point of the given circle: the part of the joining line which lies without the circle is the plane required.

Prob. IV. From a given circle to a given point without it.

Join the given point with the highest point of the given circle: the part of the joining line which lies without the circle is the plane required.

Prob. V. From a given point within a given circle to the circle.

Join the given point and the highest point of the circle: the part of the joining line produced which is between the point and the circle, is the plane required.

Prob. VI. From a given circle to a given point within it.

Join the given point and the lowest point of the circle: the part of the line produced which is between the circle and the point, is the plane required.

Prob. VII. From a given straight line (RM) without a given circle (ASB) to the circle.

Through B, the lowest point of the circle, draw BM horizontal, meeting the given line in M. Take MR upwards along the line equal to MB, and join RB meeting the circle in S: RS is the plane required.

Prob. VIII. From a given circle to a given straight line without it.

Draw a horizontal line through the highest point of the circle, terminated by the given line; and take downwards along the given line a distance equal to this horizontal line. Join the extremity of this distance with the highest point: a part of this joining line is the plane required.

Prob. IX. From a given circle to another given circle without it.

Join the highest point of the first circle with the lowest point of the second: the portion of the joining line which is between the circles, is the plane required.

Prob. X. From a given circle to another given circle within it.

Join the lowest point of the first circle with the lowest point of the second: the part of the joining line produced which lies between the two circles is the plane required.

Prob. XI. From a given circle within another given circle to the other circle.

Join the highest point of the first circle with the highest point of the second; the part of the joining line produced which lies between the two circles is the plane required.

It may also be required to find the plane of longest descent;—

PROB. XII. From a given point without a given circle to the circle.

Join the given point and the highest point of the circle: this joining line, produced till it again meets the circle, is the plane required.

PROB. XIII. From a given circle to a given point without it.

Join the given point and the lowest point of the circle: this joining line, produced till it again meets the circle, is the plane required.

PROB. XIV. From a given circle to another given circle without it.

Join the lowest point of the first circle with the highest point of the second: the joining line, produced both ways, till it again meets the circumferences, is the plane required.

The plane of longest descent cannot be determined in any case where there is a possibility of drawing a horizontal plane under the conditions: for as the plane approaches to this position, the time of descent increases without limit.

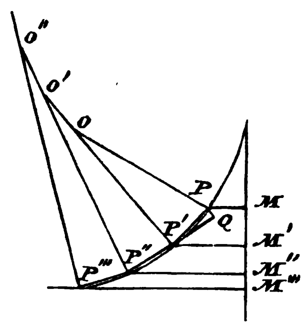
Also the plane of shortest descent cannot be determined in any case when the circles, &c. between which it is to be drawn, intersect each other: for, by bringing the extremities of the plane near this point, we may diminish the plane, and the time down it, indefinitely.

SECTION IV.

MOTION ON A CURVE.

191. PROP. *When a body descends down any curve by the force of gravity, the velocity acquired at any point will be the same as if the body had fallen freely down the vertical descent.*

A curve may be considered as the limit of a polygon when the sides become indefinitely small and indefinitely numerous. But in descending down any polygon, as $PP'P''P'''$, the velocity is increased in descending down each side, but diminished in each transit from one side to another. For the body which is moving in the direction PP' meets a plane $P'P''$ which makes an angle with PP'; and the body goes on in PP' with only a portion of its velocity.



Suppose the body to move with perfect freedom along the planes PP', $P'P''$: draw PQ perpendicular to $P'P''$ produced. Then the velocity PP' may be resolved into PQ, QP'; of which PQ is lost and QP' retained. Also $QP' = PP' \times \cos PP'Q$.

Let PO, $P'O'$, $P''O''$ be drawn perpendicular to $P'P$, $P''P'$, $P'''P''$. Therefore angle $PP'Q = POP'$. Hence velocity at P' in $P'P''$ = velocity at P' in $PP' \times \cos POP'$.

Let u, v be the velocity at P and P' on the plane PP'; let u', v' be the velocity at P' and P'' on the plane $P'P''$; let u'', v'' be the velocity at P'' and P''' on the plane $P''P'''$. Also let ω, ω', ω'' be the angles POP', $P'OP''$, $P''OP'''$; and let h, h', h'' be the portions of the vertical descent corresponding to PP', $P'P''$, $P''P'''$. Then we have, by what has been just said, $u' = v\cos\omega$, $u'' = v'\cos\omega'$.

Hence $u'^2 = v^2\cos^2\omega$. But, by Art. 187, Cor. 2,

$$v^2 = u^2 + 2gh;$$

therefore $u'^2 = (u^2 + 2gh)\cos^2\omega = u^2\cos^2\omega + 2gh\cos^2\omega$.

In like manner for u'', u'''.

Hence
$$\begin{aligned} u'^2 &= u^2\cos^2\omega + 2gh\cos^2\omega, \\ u''^2 &= u'^2\cos^2\omega' + 2gh'\cos^2\omega, \\ u'''^2 &= \;''^2\cos^2\omega'' + 2gh''\cos^2\omega''. \end{aligned}$$

Eliminating u', u'', we have

$$u'''^2 = u^2 \cos^2\omega \cos^2\omega' \cos^2\omega''$$
$$+ 2g(h\cos^2\omega\cos^2\omega'\cos^2\omega'' + h'\cos^2\omega'\cos^2\omega'' + h''\cos^2\omega'').$$

But if PP', $P'P''$, $P''P'''$ be l, l', l'' respectively, and if OP, $O'P'$, $O''P''$ be r, r', r'' respectively, we have (since $OP'P$, $O'P''P'$, $O''P'''P''$ are right angles)

$$\sin\omega = \frac{l}{r}, \ \sin\omega' = \frac{l'}{r'}, \ \sin\omega'' = \frac{l''}{r''}.$$

Hence $\cos^2\omega = 1 - \dfrac{l^2}{r^2}$, $\cos^2\omega' = 1 - \dfrac{l'^2}{r'^2}$, $\cos^2\omega'' = 1 - \dfrac{l''^2}{r''^2}$.

Therefore substituting

$$u'''^2 = u^2\left(1 - \frac{l^2}{r^2}\right)\left(1 - \frac{l'^2}{r'^2}\right)\left(1 - \frac{l''^2}{r''^2}\right)$$
$$+ 2g\left\{h\left(1 - \frac{l^2}{r^2}\right)\left(1 - \frac{l'^2}{r'^2}\right)\left(1 - \frac{l''^2}{r''^2}\right) + h'\left(1 - \frac{l'^2}{r'^2}\right)\left(1 - \frac{l''^2}{r''^2}\right)\right.$$
$$\left. + h''\left(1 - \frac{l''^2}{r''^2}\right)\right\}.$$

And expanding,

$$u'''^2 = u^2\left\{1 - \left(\frac{l^2}{r^2} + \frac{l'^2}{r'^2} + \frac{l''^2}{r''^2}\right) + \frac{l^2 l'^2}{r^2 r'^2} + \&c.\right\}$$
$$+ 2g\left\{h + h' + h'' - \left(\frac{l^2 h}{r^2} + \frac{l'^2 h'}{r'^2} + \frac{l''^2 h''}{r''^2}\right) + \frac{l^2 l'^2 h}{r^2 r'^2} + \&c.\right\}.$$

Now when we take the whole figure and the limit, $h + h' + h''$ becomes the whole vertical descent $= H$, suppose; and u''' becomes the final velocity $= U$, suppose. Also the quantity $\dfrac{l^2}{r^2} + \dfrac{l'^2}{r'^2} + \dfrac{l''^2}{r''^2}$ vanishes at the limit. For the quantities r, r', r'' are all ultimately finite, since they are ultimately the radii of curvature. And if n be the number of the parts l, l', l'', and L the whole finite arc made up of these parts, each of the parts bears a finite ratio to the mean of them, namely to $\dfrac{L}{n}$. Let m be this ratio for the greatest part;

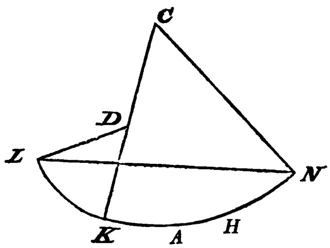
therefore the greatest of the parts l, l', l'' is $\frac{mL}{n}$. And since the others are less, the sum of n terms $\frac{l^2}{r^2} + \frac{l'^2}{r'^2} +$ &c. is less than the sum of the same number of terms, each being $\frac{m^2L^2}{n^2r^2}$; that is, less than $\frac{m^2L^2}{nr^2}$. And in this expression every thing is finite except n, which at the limit is infinite. Therefore at the limit the sum $\frac{l^2}{r^2} + \frac{l'^2}{r'^2} +$ &c. vanishes. In the same manner it may be shown that the sum $\frac{l^2h}{r^2} + \frac{l'^2h'}{r'^2} +$ &c. vanishes at the limit; and in like manner the succeeding terms of the expansion vanish. Hence we have, for the curve,

$$U^2 = u^2 + 2gH;$$

and this is the same formula which we should have if the body fell vertically down the height H; u being the initial, and U the final velocity. Therefore the velocity acquired by a body in falling through a vertical space H, beginning with the velocity u, is the same as the velocity acquired in falling along the curve through the same vertical height, beginning with the same velocity.

Cor. 1. The velocity acquired by a body falling from rest down any curve is equal to the velocity acquired by a body falling from rest through the same vertical height.

Cor. 2. If a body be projected along a curve upwards, it will lose all its velocity in ascending through the vertical height due to the velocity of projection. For in ascending it will be acted upon by the same forces as if it were descending, and therefore will lose velocity by the same degrees in the same points by which it acquired velocity in descending; and will have lost all its velocity, and be reduced to rest, when it reaches the point at which it must have begun to fall from rest, in order to acquire the velocity. That is, it will attain the vertical height due to the velocity.



COR. 3. If the curve on which the body moves be a continuous curve HAK, rising on both sides of the lowest point, a body descending from rest at H will ascend to K, HK being a horizontal line.

COR. 4. If the body hang by a string, and swing, it will describe a curve, and the motion will be the same as if it moved upon a perfectly smooth and hard curve; and therefore all the above conclusions are true in this case.

COR. 5. If a body swinging at the end of a string, descend from rest, and if the string in its ascent catch hold of an obstacle, so as to change the curve described, the body will still ascend to the same horizontal line from which it fell.

Thus, let the body appended to the string CN catch the peg D in ascending, so as to describe the arc NK with center C on one side of the line CD, and the arc KL with center D on the other side of CD; the body descending from rest at N will swing onwards to L, NL being a horizontal line.

This was observed as a fact by Galileo; and the fact serves to confirm the mechanical principles by means of which we have here proved the result.

192. DEF. If a circle roll along a straight line in its own plane, a point in the circumference describes a curve, which is called a *Cycloid*.

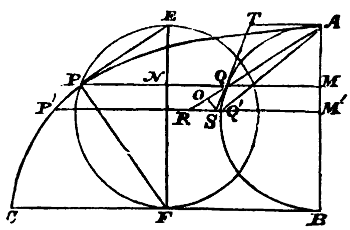
The circle which rolls is called the *generating circle*, (as EPF, fig. to next Article). The point is the *describing point*. The line on which it rolls is the *base* of the cycloid, (as CB). The *complete* cycloid is the curve described by the describing point from its leaving the base to its returning to it again, (and will consist of the arc CA, and an equal and opposite arc on the other side of AB). When the generating circle touches the base in the point diametrically opposite to the describing point, the diameter drawn through the touching point is the *axis* of the cycloid, (as AB). The extremity of

the axis (A) is the *vertex* of the curve. The generating circle when in this position, is the *axial circle* (AQB).

If, through any point in the cycloid, a line be drawn perpendicular to the axis (as PQM), this cuts the axial circle in the *corresponding point* (Q); and the portions of this line which fall within the cycloid, and the circle, (PM, QM) are the *ordinates* of the cycloid and of the circle respectively.

193. PROP. *The ordinate of the cycloid is equal to the sum of the corresponding ordinate and arc of the axial circle.*

Let EPF be the position of the generating circle at the time when the describing point is at P. Then the arc PF has been applied to CF, so that all the points of each have successively coincided: therefore the two are equal, that is, CF = arc PF = arc QB. For the same reason CB = semicircle ABQ.



Hence, taking away equals, FB = arc AQ. But evidently, $PN = QM$, and therefore $PQ = NM = FB$. Hence PQ = arc AQ; and PM = arc $AQ + QM$.

194. PROP. *The tangent to the cycloid at the point* P *is parallel to the corresponding chord of the axial circle.*

In last fig. if the circle EPF be supposed for an instant to turn round a fixed point F, instead of rolling along FB, the motion of P will be ultimately in the same direction, on either supposition. But on the former supposition, the motion of P will evidently be perpendicular to FP, or in the direction PE. Hence the direction of the curve CP at P is PE, and therefore PE is a tangent. And PE is parallel to QA: hence the tangent at P is parallel to QA.

195. PROP. *The arc of the cycloid, measured from the vertex of the curve, is equal to double the corresponding chord of the axial circle.*

(See last fig.) Let $M'Q'P'$ be an ordinate very near to MQP. Let AQ meet $M'Q'$ in R, and draw at Q a tangent

to the circle meeting $M'Q'$ in S, and meeting a tangent at A in T. Also draw SO perpendicular upon QR.

Since $TA = TQ$, angle $TAQ = TQA$. Now angle $TAQ = QRS$, and $TQA = RQS$; therefore $QRS = RQS$, and $SQ = SR$. Hence the triangles SOQ and SOR are equal; wherefore $QO = RO$, and $QR = 2\,QO$.

Now when Q' approaches indefinitely near to Q, S approaches to Q', and OS coincides ultimately with a circular arc to radius AQ' and center A. Hence QO is ultimately the excess of AQ' above AQ, or the quantity by which AQ is increased.

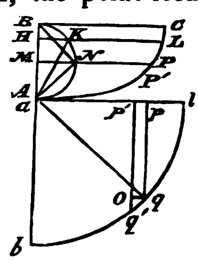
Also QR is parallel to the tangent at P, and hence QR is ultimately equal to PP', the quantity by which AP is increased.

Hence it appears, that, for corresponding points, the arc AP is ultimately increased by a quantity twice as great as the increase of the chord AQ; and therefore, as AP and AQ begin together, AP will always be twice as great as the chord AQ.

Cor. $AP = 2\,PE$.

196. Prop. *If a body descend down any arc of an inverted cycloid, and if a quadrant be described, in which the radius corresponds to the whole arc descended, the velocity at any point will be as the corresponding sine, and the time as the corresponding arc, in the quadrant.*

Let CA be the inverted semi-cycloid; L, the point from which the body descends, P any other point. Let a straight line al be taken equal to the arc AL, and in it ap equal to the arc AP; and a quadrant being described with radius al, let pq perpendicular to al meet the quadrant in q. The velocity at P is as pq, and the time of describing LP is as lq.



Draw LH, PM horizontal, meeting the axis; and let v be the velocity at P;

$$\text{then } v^2 = 2g\,.\,HM = 2g\,(AH - AM)\,;$$

$$\text{or } v^2 = 2g\left(\frac{AK^2}{AB} - \frac{AN^2}{AB}\right) = \frac{g}{2}\,\frac{4AK^2 - 4AN^2}{AB} = \frac{g}{2}\,\frac{AL^2 - AP^2}{AB}$$

$$\text{(because } AL = 2AK,\ AP = 2AN,) = \frac{g}{2}\,.\,\frac{al^2 - ap^2}{AB} = \frac{g}{2}\,.\,\frac{pq^2}{AB}\,.$$

Hence v is as pq, and pq may represent the velocity at P.

Also the time of describing a small arc PP' with the velocity v, is equal to the time of describing a corresponding small arc pp' with the velocity represented by pq. But if $p'q'$ be parallel to pq, and qo to al, we have ultimately,

$$pq : qa :: qo : qq'; \text{ hence } \frac{qq'}{qa} = \frac{qo}{pq}, \text{ or } \frac{qq'}{al} = \frac{pp'}{pq}.$$

Hence the time of describing pp' with the velocity represented by pq, is equal to the time of describing qq' with the velocity represented by al. In like manner, the time of describing any other indefinitely small arc of the cycloid with the velocity which belongs to its initial point, is equal to the time of describing the corresponding arc of the quadrant with the constant velocity al. But the limit of the sum of all the times of describing small arcs PP' with the velocity belonging to the initial point of each, is the time of describing the whole arc LP' with the variable velocity. And the sum of the times of describing small arcs qq' with a constant velocity al, is the time of describing the arc lq' with the constant velocity al. Therefore the time of P descending along LP is equal to the time of q moving along lq, with a constant velocity al; and therefore varies as lq.

Hence in the cycloid, the velocity at P is as pq, the sine; and the time in LP is as lq, the arc.

Cor. 1. As above, $v^2 = \frac{g}{2}\frac{pq^2}{AB}$; hence $v = \frac{pq\sqrt{g}}{\sqrt{2AB}}$; and this is the velocity represented by pq.

Cor. 2. Hence the velocity represented by al is $\frac{al\sqrt{g}}{\sqrt{2AB}}$.

And the time of describing $lq = \frac{\text{arc } lq}{\text{velocity } al} = \frac{lq}{al}\sqrt{\frac{2AB}{g}}$;

which is therefore the time of a body descending down LP.

Cor. 3. If $AL = l$, $AP = x$, $AB = a$, we have

$$v^2 = \frac{g}{2}\frac{AL^2 - AP^2}{AB} = \frac{g}{2a}(l^2 - x^2).$$

Cor. 4. Since angle $\frac{lq}{al}$ = angle $\left(\cos = \frac{ap}{aq} = \frac{x}{l}\right)$

$$\text{time} = \sqrt{\frac{2a}{g}} \cdot \text{arc} \left(\cos = \frac{x}{l}\right).$$

The last two corollaries agree with the expressions obtained in Article 155 for the motion of a body acted upon by a force which varies as the distance from a fixt point, putting $\frac{g}{4a}$ for μ in these expressions.

197. The force does vary in this proportion in the cycloid.

Prop. *In an inverted cycloid the force which urges a heavy body along the arc is as the distance from the lowest point.*

For the force of gravity being represented by a vertical line BA, (fig. Art. 196,) is equivalent to two forces BN, NA; of which BN is destroyed by the reaction of the curve: and hence the force along the curve is as NA: that is, as $2NA$, or as PA.

198. Prop. *The time of a body's descending from rest, from any point in the arc of an inverted cycloid to the lowest point, is the same, from whatever point of the curve the body begins to descend.*

By Art. 196. Cor. 2, the time of descending from a point L down any arc LP, is $\frac{lq}{al}\sqrt{\frac{2AB}{g}}$. And hence the time of descending to the lowest point A, is $\frac{lb}{al}\sqrt{\frac{2AB}{g}}$; where lb is the quadrant to radius al. But this being the case, the fraction $\frac{lb}{al}$ is the same, whatever be the radius al. Therefore the time of falling to A is the same whatever be the point L.

This is sometimes expressed by saying that the cycloid is the *isochronous curve*.

Cor. 1. If $\pi = 3.14159$, the ratio of the circumference of a circle to the diameter, $lb = \frac{\pi}{2} al$; whence if T be the time of descent down LA, $T = \frac{\pi}{2}\sqrt{\frac{2AB}{g}}$.

Cor. 2. If the body go on beyond A with the velocity acquired, it will ascend to a height equal to that from which it fell, losing velocity in its ascent by the same degrees as those by which it acquired velocity in its descent (Art. 191). And hence it will employ the same time in the ascent through any arc of the cycloid, as in the descent through the similar arc on the other side of the axis.

Cor. 3. Hence the whole time of ascent will be equal to the whole time of descent. And the time of the body moving from rest till it comes to rest again, will be double the time of descent. Therefore the time of a complete *trip* will be

$$\pi\sqrt{\frac{2AB}{g}}.$$

Cor. 4. When the body has lost all its velocity by ascending, and come to a state of rest, it will begin to descend by the force of gravity; and will descend to the lowest point, and ascend again to the same height as at first. It will then descend again; and so on indefinitely. A body moving in this manner is said to *oscillate*.

Cor. 5. The times of performing these successive oscillations are all equal.

Cor. 6. The space contained between the two extreme positions of the oscillating body on the two sides of the axis is the *amplitude* of the oscillation. It appears by the Proposition that in the same cycloid, the time of an oscillation is the same, whatever be its amplitude.

Cor. 7. We have supposed the body to slide along a perfectly smooth hard curve. But if a body freely suspended by a string be made to move, it will describe a curve; and the velocity and time of describing any position of this curve will be the same as they are when the body moves freely along

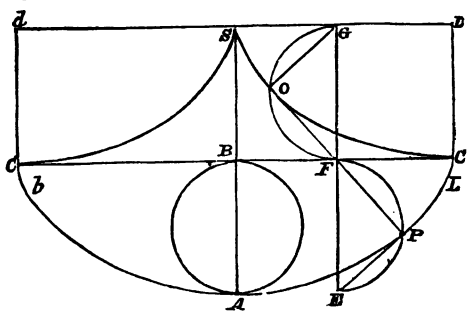
the same curve, supposed to be perfectly smooth and hard. In the following Proposition, it is proved that a body may in this way describe an inverted cycloid.

199. Prop. *To make a body oscillate in a given inverted cycloid.*

Let APC be a given semi-cycloid, AB being it axis.

Produce AB to S, making $SB = AB$: complete the rectangle $SBCD$, and with an axis CD, and base DS, describe a semi-cycloid CS.

Draw any line EFG parallel to ABS; and on opposite sides of this line describe the two equal semicircles, EPF, FOG, of the generating circles of the cycloids AC, CS. Join OF, FP. Then arc $FP = FC$, and arc $FPE = BFC$; therefore arc $PE = BF$, and $BF = SG =$ arc GO. Hence arc $PE =$ arc GO; and therefore the angles EFP, GFO are equal. Hence OFP is a straight line. Hence also $OF = FP$; therefore $OP = 2OF =$ arc OC, by Cor. to Prop. 195. And by Art. 194, OP is a tangent to the cycloid at O.



Hence it appears, that if a string SOC, fixed at S, and wrapped along the semi-cycloid SOC, be unwrapped, beginning at C, its extremity will describe a semi-cycloid CPA. And if an equal and similar semi-cycloid Sc be placed with its base Sd in the same line with DS, the same string fixed at S and wrapping upon the semi-cycloid Sc, will, with its extremity, describe the semi-cycloid Ac, thus completing the cycloid CAc. Hence a body P, suspended by a string SOP between two such semi-cycloids in a vertical plane, will oscillate in an inverted cycloid.

Cor. 1. The formulæ for the oscillations of a body moving upon a smooth hard cycloid, obtained in Articles

196, 198, apply to the case of a body oscillating as in this Proposition.

Cor. 2. If L be the whole length of the string, $L = 2AB$. Hence the time of a single oscillation is $\pi\sqrt{\frac{L}{g}}$.

A body oscillating in the manner here described is called a *cycloidal pendulum.*

200. When the amplitude of the oscillations is very small, the motion of the cycloidal pendulum will coincide very nearly with the motion of a free pendulum, the length of the string being the same. For the cycloidal *cheeks* very slightly affect the direction of the string, while it continues near to the point A; and will therefore very slightly affect the direction of the body's motion; wherefore the force which acts upon it is very slightly altered.

The free pendulum describes circular arcs in its oscillations. And we may determine all the circumstances of the *very small* oscillations of pendulums in circular arcs from the expression above,

$$t = \pi\sqrt{\frac{L}{g}},$$

where t is the time of oscillation, L the length of the pendulum.

It is manifest, from last Article, that t varies as the root of L, when g is constant:

Also that t varies inversely as the root of g, when L remains the same: and that g varies as L, when t remains the same.

Hence if λ be the length of the pendulum which oscillates seconds, and t the time in seconds of the oscillation of a pendulum whose length is L,

$$t = \sqrt{\frac{L}{\lambda}}.$$

The value of λ, the length of the seconds pendulum in the latitude of London (*in vacuo*), is found by experiment to be 39.1386 inches.

From this value of λ we can find the value of g; for making $t = 1$, we have

$$1 = \pi\sqrt{\frac{\lambda}{g}}; \quad \therefore\ g = \pi^2\lambda = 386.28 \text{ inches.}$$

Ex. 1. To find the time of oscillation of a pendulum 20 feet long,

$$t = \sqrt{\frac{240}{39.1386}} = 2.5'', \text{ nearly.}$$

Ex. 2. To find the length of a pendulum which shall make its oscillations in half minutes,

$$L = \lambda \,.\, (30)^2 = 39.1386 \times 900 \text{ inches} = 978.4 \text{ yards.}$$

201. Prop. *If a pendulum be slightly altered in length, to find the number of oscillations gained or lost in a day.*

If n be the daily number of oscillations of the pendulum in the latitude of London, (*in vacuo*,) and $N = 24 \times 60 \times 60 = 86400$, the number of seconds in 24 hours; we have

$$t = \frac{N}{n}; \quad \therefore\ \frac{N}{n} = \sqrt{\frac{L}{\lambda}}; \quad \therefore\ L = \frac{\lambda N^2}{n^2}, \quad n = N\sqrt{\frac{\lambda}{L}}.$$

Cor. If n and L be nearly equal to N and λ, we may obtain approximations for the differences.

Suppose the length of the pendulum λ to be increased by a small quantity p: to find q, the number of seconds it will lose in a day.

$$\text{Here } \lambda + p = \frac{\lambda N^2}{(N-q)^2} = \lambda\left(1 + \frac{2q}{N}\right); \text{ omitting powers of } \frac{q}{N};$$

$$\therefore\ p = \frac{2q\lambda}{N}; \quad \text{and } q = \frac{pN}{2\lambda}.$$

The same formula will apply when λ is diminished, and consequently N is increased.

Ex. A seconds pendulum is lengthened $\frac{1}{100}$ of an inch: to find the number of seconds it will lose per day.

$$\text{Here } p = .01; \quad \therefore\ q = \frac{.01 \times 86400}{2 \times 39.13} = \frac{43200}{3913} = 11'' \text{ nearly.}$$

202. PROP. *If the force of gravity be slightly altered, to find the number of seconds gained or lost in a day by a seconds pendulum.*

Let G be the value of g at a given place; when L remains the same, t varies inversely as the root of g (Art. 200); hence

$$\frac{t}{1} = \frac{\sqrt{G}}{\sqrt{g}}; \quad g = \frac{G}{t^2} = \frac{Gn^2}{N^2}.$$

COR. Hence if a seconds pendulum is taken to a place where the gravity is greater, n will be greater than N, and the pendulum will gain, and *vice versa*. The increase of gravity is generally small, and hence we may approximate as before. Let $g = G(1 + h)$, and let the pendulum gain q seconds a day, q being a small number compared with N;

$$\therefore \; G(1 + h) = \frac{G(N + q)^2}{N^2} = \frac{G(N^2 + 2qN)}{N^2}, \text{ omitting } \frac{q^2}{N^2};$$

$$\therefore \; h = \frac{2q}{N}.$$

EX. 1. A pendulum which would oscillate seconds at the equator, would, if carried to the pole, gain 5 minutes a day: to find the proportion of the polar and equatoreal gravity.

$$h = \frac{2 \times 300''}{86400} = \frac{1}{144};$$

hence gravity at the equator : gravity at the pole :: 144 : 145.

EX. 2. A pendulum which oscillates seconds, is carried to the top of a mountain whose height is m: to find the number of seconds which it will lose per day; gravity being supposed to vary inversely as the square of the distance from the center of the earth.

Let r be the distance from the center of the earth to the first station, and G the gravity at that station. Therefore $r + m$ is the distance of the second station from the center of the earth, and the gravity at that station is

$$\frac{Gr^2}{(r + m)^2} = G\left(1 - \frac{2m}{r}\right), \text{ omitting } \frac{m^2}{r^2}, \text{ \&c.}$$

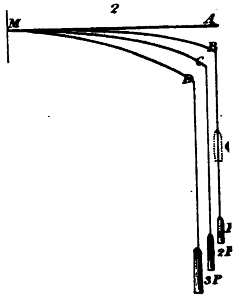
Hence, putting $\frac{2m}{r}$ for h in the formula, which will be the same for the diminution as for the increase of gravity,

$$\frac{2m}{r} = \frac{2q}{N}, \text{ and } q = \frac{Nm}{r}.$$

If the radius of the earth be 4000 miles, and the height of the mountain 1 mile,

$$q = \frac{86400}{4000} = 21''.6, \text{ the number of seconds lost per day.}$$

2



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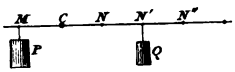

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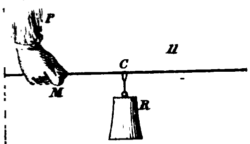

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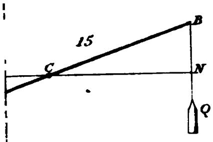

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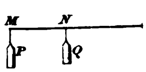

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